Total number of printed pages – 7

63 (FY)SEM–3/MAJ/MATMAJ2014

2025

MATHEMATICS

Paper : MATMAJ2014

(Elements of Real Analysis)

Full Marks : 70

Pass Marks : 28

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Choose the correct answer : 1×6=6

(a) The set $S = \{x \in \mathbb{R} : x = n+1, n \in N\}$ is

(i) Bounded above

(ii) Bounded below

(iii) Bounded

(iv) Unbounded

(b) The infimum of the set

$$\left\{1+\frac{(-1)^n}{n} : n \in N\right\} \text{ is}$$

(i) 1

(ii) 0

(iii) $\frac{3}{2}$

(iv) does not exist

(c) If a and b are any two positive real numbers such that $a<b$, then their exist a positive integer n such that

(i) $na > nb$

(ii) $na < nb$

(iii) $na \leq nb$

(iv) $na \geq nb$

(d) If $\{x_n\}$ and $\{y_n\}$ are two sequences such that $x_n \leq y_n, \forall_n \in N$. If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ then

(i) $x \leq y$

(ii) $y \leq x$

(iii) $x \geq 0$ and $y \leq 0$

(iv) $y \geq 0$ and $x \leq 0$

(e) Which one of the following is correct ?

(i) A monotonic increasing sequence is bounded

(ii) A monotonic decreasing sequence is bounded

(iii) A monotonic increasing sequence is unbounded

(iv) A monotonic increasing sequence may or may not be bounded

(f) If $\sum u_n$ is a positive series, such that

$\lim_{n\to\infty}\left(n \log \frac{u_n}{u_{n+1}}\right) = l$, then the series

(i) Converges for $l > 1$, and diverges for $l \leq 1$

(ii) Converges for $l \geq 1$, and diverges for $l < 1$

(iii) Converges for $l > 1$, and diverges for $l < 1$

(iv) Converges for $l < 1$, and diverges for $l > 1$

2. Answer the following questions : ***(any five)*** 2×5=10

(a) Prove that if $a \neq 0$ and b *in* $\mathbb{R}$ are such that $a.b=1$, then $b=1/a$ and if $a.b=0$, then ether $a=0$ or $b=0$.

(b) If $a,b \in \mathbb{R}$, then show that $|a+b| \leq |a|+|b|$.

(c) Let $a \in \mathbb{R}$. If x belongs to the neighbourhood of $V_\varepsilon(a)$ for every $\varepsilon > 0$, then show that $x=a$.

(d) Use the definition of limit of a sequence to establish the following limit.

$$\lim\left(\frac{n}{n^2+1}\right)=0$$

(e) State D'Alembert's ratio test for convergence of a series.

(f) Prove that every Cauchy sequence of real numbers is bounded.

(g) Show that the series $1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}\cdots$ is convergent.

3. Answer the following questions : ***(any six)*** 5×6=30

(a) Prove that if $x \in \mathbb{R}$, then there exists $n_x \in \mathbb{N}$ such that $x < n_x$.

(b) If x and y are real numbers with $x<y$, then show that there exist a rational number $r \in \mathbb{Q}$ such that $x<r<y$.

(c) For all $a,b,c \in \mathbb{R}$ show that $|ab|=|a||b|$ and if $c \geq 0$, then $|a| \leq c$ if and only if $-c \leq a \leq c$.

(d) Show that every convergent sequence of real numbers is bounded, but the converse is not true.

(e) Test the series for convergence

$$\frac{1.2}{3^2.4^2}+\frac{3.4}{5^2.6^2}+\frac{5.6}{7^2.8^2}+\ldots$$

(f) Find all $x \in \mathbb{R}$ that satisfy the following inequality: $|x|+|x+1|<2$.

(g) Show that for any fixed value of x, the series $\sum_{n=1}^{\infty}\frac{\sin nx}{n^2}$ is convergent.

(h) Show that the sequence $\{S_n\}$, where

$$S_n = \frac{1}{n+1} + \frac{1}{n+1} + \cdots + \frac{1}{n+n}, \forall n \in N$$ is convergent

(i) Show that the series $\sum \frac{1}{n^p}$ diverges if $p \leq 1$.

4. Answer the following questions: ***(any two)*** 12×2=24

(a) (i) If $I_n = [a_n, b_n], n \in \mathbb{N}$, is a nested sequence of closed bounded intervals, then show that there exists a number $\xi \in \mathbb{R}$ such that $\xi \in I_n$ for all $n \in \mathbb{N}$. 6

(ii) Show that the set $\mathbb{R}$ of real numbers is not countable. 6

(b) If $\sum u_n$ is a positive series, such that $\lim_{n\to\infty} (u_n)^{1/n} = l$, then prove that the series $\sum u_n$

(i) Converges, if $l < 1$,

(ii) Diverges, if $l > 1$, and

(iii) The test fails to give any definite information, if $l = 1$.

5+5+2=12

(c) State and prove Squeeze theorem. Applying Squeeze theorem, prove that

$$\lim_{n\to\infty}\left(\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots + \frac{1}{(2n)^2}\right) = 0.$$

6+6=12

(d) (i) Define monotone sequence. Prove that a monotone sequence of real numbers is convergent if and only if it is bounded. 1+5=6

(ii) Prove that a bounded sequence of real numbers has a convergent subsequence. 6