63(FY) SEM–4/MAJ/MATMAJ2034

2025

MATHEMATICS

(Major)

Paper : MATMAJ2034

(Group Theory)

Full Marks : 70

Pass Marks : 28

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Choose the correct answer : 1×6=6

(a) If 'a' and 'b' are any two elements of a group G then $(ab^{-1})^{-1}$ is

(i) $a^{-1}b^{-1}$

(ii) ba^{-1}

(iii) $a^{-1}b$

(iv) $b^{-1}a^{-1}$

(b) The number of generators of an infinite cyclic group is

(i) 1

(ii) 2

(iii) 0

(iv) infinite

(c) If G be a group of order n and H be a proper subgroup of G of order m then

(i) $m = n$

(ii) $m = nq$, q is the index of H in G

(iii) $n = mq$, q is the index of H in G.

(iv) n divides m

(d) A mapping f from a group G into a group G' is said to be homomorphism of G into G' if

(i) $f(ab)=f(a).f(b)$

(ii) $f(ab)=f(a)/f(b)$

(iii) $f(ab) = f(a)-f(b)$

(iv) $f(ab) = f(a) + f(b)$, $a, b \in G$

(e) If G is a group, then the quotient group G/H is defined if

(i) H is a subgroup of G

(ii) H is a finite subgroup of G

(iii) H is a normal subgroup of G

(iv) G is a subgroup of H

(f) In a group G, if $a^2 = e$, $\forall\ a \in G$, then G is

(i) Cyclic

(ii) Non-abelian

(iii) Abelian

(iv) Infinite

2. Answer the following question : ***(any five)*** 2×5=10

(a) Define Dihedral group. How many elements are there in D_5 ?

(b) Show that $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 1 & 2 & 6 & 7 & 5 \end{pmatrix}$ is an odd permutation.

(c) Prove that in a group G, the identity element is unique.

(d) If $f: G \to G'$ be homomorphism and e, e' are identity elements of G and G', respectively, then prove that

$$f(a^n) = [f(a)]^n,\ a \in G.$$

(e) Give reason why any Abelian group of order 15 is cyclic ?

(f) Show that every group has *at least two* normal subgroups.

(g) Justify or invalidate the following statement with argument : "A group of order 24 does possess a normal subgroup of order 14".

3. Answer the following questions : ***(any six)*** $5\times6=30$

(a) State Lagrange's theorem an order of a subgroup of a finite group. Prove that

(i) A group of prime order is cyclic.

(ii) For any element 'a' of a group of order n, $a^n = e$. $1+2+2=5$

(b) What do you mean by permutation group ? let $S = \{1, 2, 3\}$. Prove that the set S_3 of all the permutations on S is a group under multiplication of permutations. Is S_3 Abelian ? Justify your answer. $1+3+1=5$

(c) Define a normal subgroup of a group and give an example. Prove that every subgroup of an abelian group is normal. $1+1+3=5$

(d) Define a homomorphism from a group G to a group G'. Let G be a group and e be the identity element of G. Let $f: G \to G$ be defined by $f(x) = e, \forall\ x \in G$. Prove that f is a homomorphism. Is f is isomorphism ? $1+3+1=5$

(e) If G is a non-commutative group with centre Z, then show that the quotient group G/Z is non-cyclic.

(f) State and prove Cayley's theorem. $1+4=5$

(g) Define left and right cosets of a group. If H is a subgroup of G, then prove that there is a one to one correspondence between set of left coset of H in G and the set of right coset of H in G. $2+3=5$

(h) Define a subgroup of a group. Prove that a non-empty subset H of a group G is a subgroup of G if and only if

$$a, b \in H \Rightarrow ab^{-1} \in H$$ $1+4=5$

(i) If H be a normal subgroup of a group G and K, a normal subgroup of G containing H, then show that

$$G/K \cong \frac{G/H}{K/H}$$

Contd.

4. Answer the following questions : ***(any two)***

12×2=24

(a) Define automorphism and inner automorphism of a group. If $Z(G)$, $Inn(G)$ and $Aut(G)$ are respectively centre of G, inner automorphism of G and automorphism of G, then show that

$$Inn(G) \trianglelefteq Aut(G) \text{ and } G/Z(G) \cong Inn(G)$$

2+5+5=12

(b) State and prove Cauchy's theorem for Abelian group.

If G is a finite group such that $O(G) = p^2$. Where p is a prime, then prove thalt G is abelian. 2+5+5=12

(c) *(i)* Define a group.

$$\text{Let } G = \left\{ \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} : \alpha \in R \right\}$$

Prove that G is a group under matrix multiplication. 2+4=6

(ii) Prove that the intersection of *two* subgroups of a group is again a subgroup of the group.

Give an example to show that the union of *two* subgroups of a group is not a subgroup. 4+2=6

(d) *(i)* If H and K are finite subgroups of G of order $O(H)$ and $O(K)$, respectively then prove that

$$O(HK) = \frac{O(H)O(K)}{O(H \cap K)}$$ 6

(ii) Let G be a group such that $(ab)^2 = (ba)^2$, $\forall\ a, b \in G$. Suppose G also has the property that $x^2 = e, x \in G \Rightarrow x = e$. Prove that G is abelian. 6