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63/1 (SEM-4) CC10/MATHC4106

2025

MATHEMATICS

Paper : MAT-HC-4106

(Ring Theory and Linear Algebra - I)

Full Marks : 80

Pass Marks : 32

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Choose the correct answer from the following : **(any six)** 1×6=6

(a) Let $R[x]$ be the ring of polynomials of ring R and let

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m \text{ and}$$

$$g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n$$

If $f(x) + g(x) \neq 0$, then

(i) $\deg(f(x) + g(x)) \leq \max(m, n)$

(ii) $\deg(f(x) + g(x)) \geq m + n$

(iii) $\deg(f(x) + g(x)) = m + n$

(iv) $\deg(f(x) + g(x)) \geq \max(m, n)$

(b) Let S and T be two subrings of a ring R such that

$S + T = \{s + t : s \in S, t \in T\}$, then

(i) $S + T$ is not a subset of R

(ii) $S + T$ is not a subring of R

(iii) $S + T$ is a subring of R

(iv) $S + T$ may not be a subring of R

(c) Let R and S be two rings such that

$\text{Ch}(R \times S) = 0$, then

(i) $\text{Ch}(R) \neq 0$ and $\text{Ch}(S) \neq 0$

(ii) $\text{Ch}(R) = 0$ and $\text{Ch}(S) \neq 0$

(iii) $\text{Ch}(R) \neq 0$ and $\text{Ch}(S) = 0$

(iv) $\text{Ch}(R) = 0$ and $\text{Ch}(S) = 0$

where Characteristic of $R = \text{Ch}(R)$.

(d) Let f is a homomorphism of a ring R in to a ring R' and $S = \ker f$. Then

(i) S is not an ideal of R

(ii) S is an ideal of R

(iii) S is an ideal of R'

(iv) S is a subset of R'

(e) Let R be a ring of 2×2 matrices over

integers. Let $A = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{Z} \right\}$, then

(i) A is an ideal of R

(ii) A is a left ideal of R

(iii) A is right ideal of R

(iv) A is not a subset of R

(f) Let us consider a vector space $\mathbb{R}^2(R)$,

and $\omega_1 = \{(a, 0) : a \in R\}$

$\omega_2 = \{(0, b) : b \in R\}$, then

(i) ω_1 is a vector subspace of \mathbb{R}^2 and ω_2 is not a vector subspace of \mathbb{R}^2

(ii) ω_1 is not a vector subspace of \mathbb{R}^2 and ω_2 is a vector subspace of \mathbb{R}^2

(iii) $\omega_1 \cup \omega_2$ is a vector subspace of \mathbb{R}^2

(iv) $\omega_1 \cap \omega_2$ is a vector subspace of \mathbb{R}^2

(g) Let ω be a vector subspace of vector space $\mathbb{R}^3(R)$, then

(i) $\dim \omega = 0$ is not possible

(ii) $\dim \omega = 1$ is not possible

(iii) $\dim \omega = 2$ is not possible

(iv) $\dim \omega = 4$ is not possible

(h) Let M be an n -square invertible matrix, then

(i) $\text{Rank}(M) = n - 1$

(ii) $\text{Rank}(M) = n$

(iii) $\text{Rank}(M) = n + 1$

(iv) $\text{Rank}(M) = n^2$

(i) Which of the following is true ?

(i) Every subspace of a vector space contains the zero element.

(ii) A subspace of a vector space may not contain the zero element.

(iii) A subset of a vector space that contains the zero element is linearly independent

(iv) The zero element of a vector space cannot generate a subspace of the vector space.

(j) Let us consider the set of non-zero vectors $\{v_1, v_2, v_3, \dots, v_m\}$ of a vector space and the set is linearly dependent. Then

(i) one of them say v_1 is a linear combination of preceding vectors.

(ii) two of the vectors say $v_1, v_2, v_3, \dots, v_m$ are equal.

(iii) the set $\{v_1, v_2, v_3, \dots, v_m\}$ is superset of a linearly dependent set.

(iv) the set $\{v_1, v_2, v_3, \dots, v_m\}$ is a subset of a linearly dependent set.

2. Answer the following questions : **(any five)**

$2 \times 5 = 10$

(a) Prove that a field is an integral domain.

(b) Show that the subring $\langle Z, +, \cdot \rangle$ of the ring $\langle Q, +, \cdot \rangle$ is not an ideal .

(c) If A is an ideal of a ring R with unity such that $I \in A$, then show that $A = R$.

(d) If $f: R \rightarrow R'$ be a homomorphism, then prove that

(i) $f(0) = 0'$

(ii) $f(-a) = -f(a)$

Where $a \in R$ and $0, 0'$ are zeros of rings R and R' respectively.

(e) If S_1 and S_2 be two subsets of a vector space V , then prove that

$$L(S_1 \cup S_2) = L(S_1) + L(S_2).$$

(f) Show that :

$\{(1, 2, 4), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a linearly dependent subset of the vector space $V_3(R)$.

(g) Let T be a linear operator of R^3 defined by

$$T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3)$$

Show that T is invertible.

3. Answer the following questions : **(any six)**
5×6=30

(a) Prove that a ring R is without zero divisors if and only if the cancellation laws hold in R .

(b) If D is an integral domain, then prove that characteristic of D is either zero or a prime number.

(c) Let $A \subseteq B$ be two ideals of a ring R . Prove that

$$\frac{R}{B} \cong \frac{R/A}{B/A}$$

(d) Let R be a commutative ring. Prove that an ideal P of R is a prime ideal if and only if for two ideals A, B of R , $AB \subseteq P \Rightarrow$ either $A \subseteq P$ or $B \subseteq P$.

(e) A non-empty W subset of a vector space $V(F)$ is a subspace of V if and only if for each pair of vectors α, β in ω and each scalar a in F , the vector $a\alpha + \beta$ is again in ω .

(f) Let $V(F)$ be a vector space. If $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ are non-zero vector $\in V$, then prove that either they are linearly independent or some $\alpha_k, 2 \leq k \leq n$, is a linear combination of the preceding ones $\alpha_1, \alpha_2, \dots, \alpha_{k-1}$.

(g) Prove that every n -dimensional vector space $V(F)$ is isomorphic to $V_n(F)$.

- (h) Prove that every linearly independent subset of a finitely generated vector space $V(F)$ is either a basis of V or can be extended to form a basis of V .
- (i) Define kernel of a homomorphism of a vector space $u(F)$ into a vector space $V(F)$. Also show that kernel of homomorphism is a subspace of $V(F)$.
- (j) Let S, T be two linear transformation from $V(F)$ into $V(F)$. Let β be an ordered basis of V . Then prove that $[ST]_{\beta} = [S]_{\beta} [T]_{\beta}$.

4. Answer the following questions : **(any two)**
 $10 \times 2 = 20$

- (a) (i) Let R be a ring. Prove that $a^2 - b^2 = (a+b)(a-b)$ for all $a, b \in R$ if and only if R is commutative. 5

(ii) Show that the set

$$S = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in Z \right\}$$

of diagonal matrices is a subring of the ring of all 2×2 matrices over Z . 5

- (b) (i) Let v be a vector space of ordered pairs of complex numbers over the real field R . Then show that the set

$$S = \{(1, 0), (i, 0), (0, 1), (0, i)\}$$
 is a basis of $V(R)$. 6

- (ii) Show that the ring $M_2(Z)$ of 2×2 matrices over the integers have zero divisor. 4

- (c) Define rank, nullity and dimension of a vector space $V(F)$. If T be a linear transformation on an n -dimensional vector space $V(F)$, then prove that

$$\text{Rank}(T) + \text{Nullity}(T) = \dim(V) \quad 3+7=10$$

- (d) Let T be a linear operator on F^n and A be the matrix of T the standard basis for F^n . Let ω be the subspace of F^n spanned by the column vectors of A . Find the relation between ω and T

5. Answer the following questions : **(any one)**
 $14 \times 1 = 14$

- (a) (i) Prove that if K is any field which contains integral domain D , then K contains a subfield isomorphic to F , where F is the field of quotients of the integral domain D . 6

- (ii) Find the field of quotients of the integral domain

$$\mathbb{Z}[i] = \{a + ib : a, b \in \mathbb{Z}\} \quad 8$$

- (b) If A and B be two ideals of a ring R , then prove that

(i) $A + B$ is an ideal of R containing both A and B . 6

(ii) AB is an ideal of R 5

(iii) $A + B$ is the smallest ideal containing $A \cup B$. 3

- (c) (i) Prove that *any two* bases of a finite dimensional vector space V have same number of elements. 10

(ii) Find the linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ whose range is generated by

$(1, 0, -1)$ and $(1, 2, 2)$. 4