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63/1 (SEM–4) CC9/MATHC4096

2025

MATHEMATICS

Paper : MATHC4096

(Riemann Integration and Series of Functions)

Full Marks : 80

Pass Marks : 32

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Choose the correct answer ***(any six)*** :

$1 \times 6 = 6$

(i) If f is a continuous function on $[a, b]$ that is differentiable on (a, b) and if f' is integrable on $[a, b]$, then

(a) $\int_a^b f' = f(b) + f(a)$

(b) $\int_a^b f' = f(b) - f(a)$

(c) $\int_a^b f'/f = f(a) - f(b)$

(d) $\int_a^b f'/f = 0$

(ii) Suppose $a < b$, and for all $a < x < b$, $0 \le f(x) \le g(x)$, and f, g are integrable over $[a, x]$. Then which of the following is false?

(a) If $\int_a^b g$ coverges, then so does $\int_a^b f$.

(b) If $\int_a^b f$ diverges, then so does $\int_a^b g$.

(c) If $\int_a^b g$ diverges, then so does $\int_a^b f$.

(d) If $\int_a^b g$ converges, then so does $\int_a^b f$ and $\int_a^b f \le \int_a^b g$.

(iii) Let f be integrable on $[a, b]$. Then which of the followibng is false?

(a) $\lim_{c \to a^+} \int_c^b f = \lim_{c \to b^-} \int_a^c f$

(b) $\lim_{c \to a^+} \int_c^b f = \int_a^b f$

(c) $\lim_{c \to b^-} \int_a^c f = \int_a^b f$

(d) $\lim_{c \to a^+} \int_c^b f \ne \lim_{c \to b^-} \int_a^c f$

(iv) Which of the following is correct?

(a) $\int_1^\infty 1/x^2 \, dx$ is divergent and $\int_1^\infty 1/x^2 \, dx = \infty$

(b) $\int_1^\infty 1/x^2 \, dx$ is divergent and $\int_1^\infty 1/x^2 \, dx = -\infty$

(c) $\int_1^\infty 1/x^2 \, dx$ is convergent and $\int_1^\infty 1/x^2 \, dx = -1$

(d) $\int_1^\infty 1/x^2 \, dx$ is convergent and $\int_1^\infty 1/x^2 \, dx = 1$

(v) Choose the correct statement of the following :

(a) $\int_1^\infty 1/\sqrt{x}\, dx$ is convergent and $\int_1^\infty 1/\sqrt{x}\, dx = 1$

(b) $\int_1^\infty 1/\sqrt{x}\, dx$ is convergent and $\int_1^\infty 1/\sqrt{x}\, dx = -1$

(c) $\int_1^\infty 1/\sqrt{x}\, dx$ is divergent and $\int_1^\infty 1/\sqrt{x}\, dx = +\infty$

(d) $\int_1^\infty 1/\sqrt{x}\, dx$ is divergent and $\int_1^\infty 1/\sqrt{x}\, dx = -\infty$

(vi) $\lim_{n\to\infty} (x/n) = ?$ (x is any real number)

(a) 0

(b) $+\infty$

(c) $-\infty$

(d) x

(vii) $\lim_{n\to\infty} \left(\frac{x}{x+n}\right) = ?$

(x is a real number such that $x \geq 0$)

(a) 0

(b) $+\infty$

(c) $-\infty$

(d) 1

(viii) If radius of convergence of $\sum_{n=0}^{\infty} x^n$ is 1, then the radius of convergence of $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$ is

(a) 1

(b) 2

(c) 3

(d) 4

(ix) Let $f_n(x) = n^2x^n(1-x)\ \forall\ x \in [0,1]$. Then $\lim_{x\to\infty} f_n(1)$ is

(a) -1

(b) 1

(c) 0

(d) ∞

(x) $\lim_{n\to\infty}\left\{\left(x^2+nx\right)/n\right\} = ?$ (x is any real number)

(a) x

(b) $-x$

(c) 0

(d) ∞

2. Answer the following questions ***(any five)***: 2×5=10

(a) Define a mesh (or a norm) of a partition of an interval $[a,b]$.

(b) Let f be a bounded function on $[a,b]$. Define a Riemann sum of f associated with a partition P of $[a,b]$.

(c) suppose that $a > 0$ and that f is integrable on $[-a, a]$. If f is bounded, show that $\int_{-a}^{a} f = 0$.

(d) Show that $\lim_{n\to\infty}\left(nx/\left(1+n^2x^2\right)\right) = 0$ $\forall\ x \in R$.

(e) Write the statement of Weierstrass's approximation theorem.

(f) State Weierstrass's M-Test for the convergence of a series of functions.

(g) Define uniform convergence of sequence of functions defined on a set $S \subseteq \mathbb{R}$.

3. Answer the following questions ***(any six)*** : 5×6=30

(i) Prove that a bounded function f on $[a, b]$ is integrable if and only if for each $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that $U(f,P) - L(f,P) < \varepsilon$.

(ii) Prove that every continuous function f on $[a, b]$ is integrable.

(iii) Show that $\int_0^1 1/\sqrt{x}\, dx$ is an improper integral and determine its convergence or divergence. $2\frac{1}{2} + 2\frac{1}{2} = 5$

(iv) Let f be an integrable function on $[a, b]$. For x in $[a, b]$, let $F(x) = \int_a^x f(t)dt$. Then prove that F is continuous on $[a,b]$. Is F differentiable in (a, b) ? Justify your answer. 3+2=5

(v) Let (f_n) be a sequence of continuous functions on $[a, b]$ and suppose that $(f_n \to f)$ uniformly on $[a, b]$. Then prove that $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$

(vi) Show that $\sum_{n=1}^{\infty} \left(\frac{x^n}{2^n}\right)$ represents a continuous function f on $(-2, 2)$.

(vii) Suppose that f and g are continuous functions on $[a, b]$ such that $\int_a^b f = \int_a^b g$. Prove that there exists an $x \in (a, b)$ such that $f(x) = g(x)$.

(viii) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ have radius of convergence $R > 0$. Then prove that f is differentiable on $(-R, R)$ and $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ for $(x) < R$.

(ix) Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Then prove that

$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} \left(\frac{a_n}{n+1} x^{n+1}\right) \text{ for } |x| < R.$$

(x) Show that if function f defined on $[a, b]$ is integrable on $[a, b]$, then $|f|$ is integrable on $[a, b]$ and $\left|\int_a^b f\right| \le \int_a^b |f|$.

3+2=5

4. Answer the following questions ***(any two)*** : 10×2=20

(i) *(a)* Let

$$f(x) = \begin{cases} x & \text{for } x \text{ is rational} \\ 0 & \text{for } x \text{ is irrational} \end{cases}$$

Calculate the upper and lower Darboux integrals for f on the interval $[a, b]$. Is f integrable on $[a, b]$? 3+3+1=7

(b) If f and g are integrable on $[a,b]$ and if $f(x) \le g(x) \forall x \in [a, b]$, then prove that $\int_a^b f \le \int_a^b g$ 3

(ii) Let f and g be integrable on $[a, b]$, and let c be a real number. Then prove that

(a) cf is integrable and $\int_a^b cf = c\int_a^b f$ 5

(b) $f+g$ is integrable and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$ 5

(iii) When is a sequence of functions defined on a set $S \subseteq \mathbb{R}$ said to be uniformly Cauchy on S? Let (f_n) be a sequence of functions defined and uniformly Cauchy on a set $S \subseteq \mathbb{R}$. Then prove that there exists a function f on S such that $f_n \to f$ uniformly on S. 2+8=10

(iv) Discuss the convergence and divergence of Gamma function

$$\int_0^\infty t^{n-1}e^{-t}dt \text{ for all } n \in \mathbb{R}$$

5. Answer the following questions ***(any one)*** : 14×1=14

(i) (a) If f is a piecewise continuous function or a bounded piecewise monotone function on $[a, b]$, then prove that f is integrable on $[a, b]$. 7

(b) Show that if f is integrable on $[a, b]$, then f^2 is also integrable on $[a, b]$ 7

(ii) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with finite positive radius of convergence R. If the series converges at $x = R$, then prove that f is continuous at $x = R$, and if the series converges at $x = -R$, then prove that f is continuous at $x = -R$. Hence, discuss the continuity of $f(x)$ at $x = -1$ and 1 for the function

$$f(x) = \sum_{n=0}^{\infty} x^n \text{ for } |x| < 1.$$ 10+4=14

(iii) (a) If f is a continuous function on $[a, b]$, then prove that there exists at least one $x \in (a,b)$ such that

$$f(x) = \frac{1}{b-a}\int_a^b f.$$ 6

(b) Examine the convergence of

$$\int_0^\infty \sin^2 \tfrac{x}{2} \Big/ x^2 \, dx$$ 4

(c) Test the uniform convergence of the series

$$\frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \ldots,$$

$$-\tfrac{1}{2} \le x \le \tfrac{1}{2}.$$ 4