Total number of printed pages–8

63 (FY)SEM–2/MAJ2/MATMAJ1024

2024

MATHEMATICS

Paper : MATMAJ1024

(Calculus)

Full Marks : 70

Pass Marks : 28

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Choose the correct answer : 1×8=8

   (a) $\lim_{x \to 0} x \sin\frac{1}{x}$ is equal to

   (i) 1

   (ii) ∞

   (iii) 0

   (iv) oscillatory

Contd.

(b) A function $f(x)$ is said to be continuous at $x = a$ if

(i) $\lim_{x\to a} f(x)$ exists

(ii) $f(a)$ exists

(iii) $\lim_{x\to a} f(x) = f(a)$

(iv) $\lim_{x\to a} f(x) \neq f(a)$

(c) If $y = x^n$, then $D^n(y) =$

(i) $n(n-1)$

(ii) $\underline{|n}$

(iii) $\underline{|m-n}$

(iv) 0

(d) The $(n+1)$th term in the expansion of $f(a+h)$ in Taylor's series expansion is

(i) $\frac{1}{\underline{|n}} f^n(a)$

(ii) $\frac{h}{\underline{|n}} f^n(a)$

(iii) $\frac{h^n}{\underline{|n}} f^n(a)$

(iv) $h^n f^n(a)$

(e) $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx =$

(i) π

(ii) $\pi/4$

(iii) $\pi/2$

(iv) 2π

(f) The transformation from Cartesian to polar coordinates in x-direction is given by

(i) $x = r \sin\theta$

(ii) $x = r \cos\theta$

(iii) $x = r \cos\theta/2$

(iv) $x = r \cos 2\theta$

(g) Let f be a function defined on $[a, b]$ such that $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b). Then which of the following is true ?

(i) There exists a point $c\in(a, b)$ at which the tangent on the curve is parallel to the x-axis

(ii) There exists a point $c\in(a, b)$ at which the tangent on the curve is parallel to the secant joining end points of $[a, b]$

(iii) There exists a point $c \in (a, b)$ at which the tangent on the curve is parallel to the *J*-axis

(iv) There exists a point $c \in (a, b)$ at which the tangent on the curve is perpendicular to the secant joining the end points of $[a, b]$

(h) The radius of curvature *P* of any point (x, y) on the curve $y = f(x)$ is

(i) $\frac{(1+y_1)^{3/2}}{y_1^2}$

(ii) $\frac{(1+y_2)^{3/2}}{y_1^2}$

(iii) $\frac{\left(1+y_1^2\right)^{3/2}}{y_2}$

(iv) $\frac{\left(1+y_2^2\right)^{3/2}}{y_1^2}$

2. Answer the following questions (***any six***) : 2×6=12

(a) Evaluate : $\int_0^{\pi/2} \sin^6 x dx$.

(b) Prove that if *f* is finitely derivable at *C*, then *f* is also continuous at *C*.

(c) Give the geometrical interpretation of Rolle's theorem.

(d) Find $\frac{dy}{dx}$, if $y = \tan hx$.

(e) Evaluate : $\int_0^{\pi/2} \sin^6 x \cos^2 x \, dx$.

(f) Write down the formula to find the length of the perpendicular from pole to the tangent to the curve $r = f(\theta)$ at $P(r, \theta)$.

(g) Show that the pedal equation of

$$r^2 = a^2 \cos 2\theta \text{ is } r^3 = a^2 p.$$

(h) Apply ε, δ definition of limit to prove that $\lim_{x \to 4} (2x-2) = 6$.

3. Answer the following questions (***any six***) : 5×6=30

(a) If a function *f* is defined by

$$f(x) = \begin{cases} \frac{xe^{1/x}}{1+e^{1/x}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

show that *f* is continuous at $x = 0$.

(b) If $y = a\cos(\log x) + b\sin(\log x)$, then show that,

(i) $x^2y_2 + xy_1 + y = 0$

(ii) $x^2y_{n+2} + (2n+1)x\, y_{n+1}+(n^2+1)y_n = 0$

2+3=5

(c) Find a reduction formula for the integral $\int \cos^n x\, dx$.

(d) Find the angle of intersection of the cardioides

$$r = a(1+\cos\theta),\ r = e(1-\cos\theta).$$

(e) Using Maclaurin's theorem, expand $\log(1+x)$ as a power series in x.

(f) The cardioid $r = a(1-\cos\theta)$ revolves about initial line. Show that the volume generated is $\frac{8}{3}\pi a^3$.

(g) Evaluate : *(any one)*

(i) $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$

(ii) $\lim_{x\to 0}\frac{\tan x - x}{x - \sin x}$

(h) Let $f(x) = e^x$ and $F(x) = e^{-x}$ be functions defined on $[a, b]$. Show that f and F satisfy the conditions of Cauchy Mean Value Theorem. Hence find a point $c \in (a,b)$ to show that it is the arithmetic mean between a and b. 2+3=5

(i) Using Lagrange's mean value theorem show that

$$\frac{x}{1+x} < \log(1+x) < x \qquad x > 0$$

Hence show that

$$0 < [\log(1+x)]^{-1} - x^{-1} < 1 \quad \forall x > 0.$$

4+1=5

4. Answer the following questions (***any two***) : 10×2=20

(a) (i) Prove that : 5

$$\int_{-a}^{a} f(x)\,d_x = \begin{cases} 0 \text{ if } f(x) \text{ is an odd function of } x \\ 2\int_0^a f(x)dx \text{ if } f(x) \text{ is an even function of } x \end{cases}$$

(ii) Define asymptotes. Find the asymptotes of the curve

$$(x^2 + y^2)x - ay^2 = 0$$

parallel to x and y axes. 5

(b) (i) State and prove Rolle's theorem. 5

(ii) Prove that $\int_0^{\pi/2} \log \sin x \, dx = -\frac{\pi}{2} \log 2$. 5

(c) (i) If $y = \sin ax + \cos ax$, prove that $y_n = a^n\{1+(-1)^n \sin 2ax\}^{1/2}$. 5

(ii) Find the length of the are of the parabola $x^2 = 4ay$ measured from the vertex to one extremity of the latus rectum. 5

(d) (i) Show that the radius of curvature at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ of the folium $x^3+y^3 = 3axy$ is $\frac{3a}{8\sqrt{2}}$. 5

(ii) Find the Taylor's expansion of $f(a+h)$ in ascending integral powers of h with the Lagrange's form of remainder, stating clearly the various assumptions made about $f(x)$. 5