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63/1 (SEM–6) DSE1B/MATRE6026

2024

MATHEMATICS

Paper : MATRE6026

(Complex Analysis)

Full Marks : 80

Pass Marks : 32

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Choose the correct answer : ***(any six)*** 1×6=6

   (a) Multiplicative inverse of $2-3i$ is

   (i) $\frac{1}{13}(2+3i)$

   (ii) $\frac{1}{13}(2-3i)$

   (iii) $\frac{1}{13}(3+2i)$

   (iv) $\frac{1}{13}(3-2i)$

Contd.

(b) If z_1 and z_2 be two complex numbers, then which one of the following is not correct?

(i) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(ii) $|z_1 + z_2| \le |z_1| + |z_2|$

(iii) $amp\left(\frac{z_1}{z_2}\right) = amp\, z_1 + amp\, z_2$

(iv) $|z_1 z_2| = |z_1||z_2|$

(c) The modulus of $\frac{1+2i}{1-3i}$ is

(i) $\sqrt{2}$

(ii) $\frac{1}{\sqrt{2}}$

(iii) $\frac{1}{2\sqrt{2}}$

(iv) $\frac{2}{\sqrt{3}}$

(d) The value of $e^{i\pi/2}$ is

(i) $-i$

(ii) i

(iii) 1

(iv) 0

(e) The value of $\underset{z \to i}{Lt} \frac{iz^3 - 1}{z+1}$ is

(i) 2

(ii) -1

(iii) 0

(iv) ∞

(f) A non-zero number λ is said to be period of $f(z)$, if

(i) $f(z+\lambda) = f(\lambda)$

(ii) $f(z+\lambda) = f(z)$

(iii) $f(z+\lambda) = z+\lambda$

(iv) $f(z+\lambda) = z$

(g) The polar form of Cauchy-Riemann equations of $f(z) = u + iv$, where f is analytic, are

(i) $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}; \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

(ii) $\frac{\partial u}{\partial r} = r\frac{\partial v}{\partial \theta}; \frac{\partial v}{\partial r} = -r\frac{\partial u}{\partial \theta}$

(iii) $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}; \frac{\partial v}{\partial r} = r\frac{\partial u}{\partial \theta}$

(iv) $\frac{\partial u}{\partial r} = r\frac{\partial v}{\partial \theta}; \frac{\partial v}{\partial r} = \frac{1}{r}\frac{\partial u}{\partial \theta}$

(h) If poles of $f(z)$ lies outside the contour C, then

(i) $\int_C f(z)dz = 1$

(ii) $\int_C f(z)dz = 0$

(iii) $\int_C f(z)dz = 2\pi i$

(iv) $\int_C f(z)dz = \pi i$

(i) A simple closed curve is called

(i) Cauchy curve

(ii) Gauss curve

(iii) Jordan curve

(iv) Riemann curve

(j) The Taylor series expansion of $f(z) = e^z$ in the region $|z| < \infty$ is

(i) $\sum_{n=0}^{\infty} \frac{z^n}{\lfloor n}$

(ii) $\sum_{n=0}^{\infty} \lfloor n \, z^n$

(iii) $\sum_{n=1}^{\infty} \frac{z^n}{\lfloor n}$

(iv) $\sum_{n=1}^{\infty} \lfloor n \, z^n$

2. Answer ***any five*** of the following questions : 2×5=10

(a) Show that $arg\, z + arg\, \bar{z} = 2n\pi$, where n is an integer.

Contd.

(b) If $(x+iy)^3 = u+iv$, then show that

$$\frac{u}{x}+\frac{v}{y}=4\left(x^2-y^2\right).$$

(c) Show that $\underset{z\to -i}{Lt}\ \frac{iz^3+1}{z^2+1}=\frac{3}{2}$.

(d) Evaluate $\int_0^{\pi/4} e^{-2it}\,dt$.

(e) What is the condition for convergence of a power series?

(f) Find the value $sin(\pi/4+i)$.

(g) Show that, if $\underset{n\to\infty}{Lt}\ z_n = z_0$, then

$$\underset{n\to\infty}{Lt}\ |z_n| = |z_0|.$$

3. Answer ***any six*** of the following questions : 5×6=30

(a) If $u=e^x(x\cos y - y\sin y)$, find the analytic function $u+iv$.

(b) Show that an analytic function cannot have a constant absolute value without reducing it to a constant.

(c) If u and v be the real and imaginary components of $f(z)$ defined by

$$f(z)=\begin{cases} (\bar{z})^2/z & , \ z\neq 0 \\ 0 & , \ z=0 \end{cases}$$

then verify that Cauchy-Riemann equations are satisfied at $z=(0,0)$ though $f'(0)$ does not exist.

(d) Determine $f(z)=u+iv$ by obtaining a harmonic conjugate of a given harmonic function

$$u(x,y)=y^3-3x^2y.$$

(e) Evaluate $\int_C \frac{1}{z^2-8z+1}\,dz$, where C be the circle $|z|=1$ oriented counter-clockwise.

(f) Prove that the sequence $\{z_n\}$ converges to z_0, where $z_n = x_n + iy_n$ and $z_0 = x_0 + iy_0$ if and only if the real sequences $\{x_n\}$ and $\{y_n\}$ converge to x_0 and y_0 respectively.

(g) If $f(z)$ be single-valued analytic function in a simply connected domain D, if $a, b \in D$, then $\int_a^b f(z)dz = F(b) - F(a)$ where $F(z)$ is any indefinite integral of $f(z)$.

(h) Find a power series expansion of the function $f(z) = \frac{1}{3-z}$ about the point $4i$.

(i) Find Taylor's series of $(\cos z)^2$ at $z = \pi$.

(j) If $\lim_{z \to z_0} f(z)$ exists, prove that it must be unique.

4. Answer ***any two*** of the following questions : 10×2=20

(a) Find Laurent series of $f(z) = \frac{z^2}{z^2+3z+2}$ in each of the following domains :

(i) $1 < |z| < 2$

(ii) $1 < |z-3| < 2$ 5+5=10

(b) If the function $f(z)$ is analytic when $|z| < R$ and has Taylor's expansion $\sum_{n=0}^{\infty} a_n z_n$, then show that for $r < R$ we have

$$\frac{1}{2\pi}\int_0^{2\pi} \left| f\left(re^{i\theta}\right)\right|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 \, r^2$$

(c) State and prove Liouville's theorem. 2+8=10

(d) Define a Harmonic function. If $f(z)$ and $g(z)$ be both analytic in a domain D and if

$$f(z)\,g(z) = U(x, y) + iV(x, y),\ z = x + iy,$$

then prove that U and V both are harmonic in D. 2+8=10

5. Answer ***any one*** of the following questions : 14

(a) State and prove Cauchy-Goursat theorem. 2+12=14

(b) State and prove Taylor's theorem. 2+12=14

(c) (i) If $f(z)$ be an analytic function in a simply connected domain D enclosed by a rectifiable Jordan curve C and let $f(z)$ be continuous on C, then show that

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\,dz,$$

where z_0 is any point of D. 7

(ii) If $f(z)$ be single valued continuously differentiable function on a simply connected region D, i.e., $f'(z)$ exists and is continuous at each point of D, then prove that

$$\int_C f(z)\,dz = 0$$

where C is any closed contour contained in D. 7
