Total number of printed pages = 9

63/1(SEM-5) CC12/MATHC 5126

2024

MATHEMATICS

Paper : MATHC 5126

(Group Theory-II)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions.

1. Choose the correct answer : ***(any six)*** 1×6=6

   (a) If a finite group G is isomorphic to the symmetric group 'S_n', then the order of the group is

   (i) $(n-1)!$

   (ii) $(n+1)!$

   (iii) $n!$

   (iv) $(2n)$

Contd.

(b) The group of automorphisms of an infinite cyclic group is of order

(i) 2

(ii) 0

(iii) 1

(iv) ∞

(c) Let $G = \{\neq 1, \neq i, \neq j, \neq k\}$. Then the class equation of G is

(i) 8=1+1+3+3

(ii) 8=2+2+2+2

(iii) 8=4+4

(iv) 8=1+2+3+2

(d) If A and B are finite cyclic groups of order m and n respectively, then

(i) $A \oplus B$ is always cyclic

(ii) $A \oplus B$ is cyclic if and only if m and n relatively Prime

(iii) $A \oplus B$ is cyclic if and only if m and n one Prime

(iv) $A \oplus B$ is are necessarily cyclic

(e) Consider the elements (3,7) and (7,9) in U(8) $\oplus$ U(10). Then the product of (3,7) and (7,9) with respect to component wise multiplication is

(i) (5,3)

(ii) (21,63)

(iii) (5,7)

(iv) (2,3)

(f) Let $*: G \times A \to A$ be a group action and $a \in A$ then $Ker(*)$ is

(i) $\{g \in G \mid g * a = a\}$

(ii) $\{g \in G \mid g * a = e\}$

(iii) $\{g \in G \mid g * a \neq a\}$

(iv) $\{g \in G \mid g * a \neq e\}$

(g) If G' is a commutator subgroup of a group G, then $\frac{G}{G'}$ is

(i) cyclic
(ii) non cyclic
(iii) abelian
(iv) non abelian

(h) For $G = S_3$, is commutator of G then

(i) $G' = Z(G)$

(ii) $G' = S_3$

(iii) $G' = A_3$

(iv) $G' = \{e\}$

(i) The conjugate of the element $\sigma = (23)(589)$ in Sq is

(i) (13) (47) (89)

(ii) (235) (89)

(iii) (25) (1679)

(iv) (123) (567)

(j) Let G be a group acting on a non-empty set A under the group action '*'. Then

(i) $(g_1 * g_2) * a = (g_1 g_2) * a \ \forall g_1, g_2 \in G$ and $a \in A$

(ii) $g * a_1 = g * a_2 \Rightarrow a_1 = a_2 \forall g \in G$ and $a_1, a_2 \in A$

(iii) $g_1 * a = g_2 * a \Rightarrow g_1 = g_2 \forall g_1, g_2 \in G$

and $a \in A$

(iv) $e * a \neq a$ for some $a \in A$.

2. Answer the following questions : ***(any five)*** 2×5=10

(a) Show that the function $f : \mathbb{Z} \to \mathbb{Z}$ such $f(x) = -x \forall x \in \mathbb{Z}$ that is an automorphism of the additive group of ingegers $\mathbb{Z}$

(b) Define Inner Automorphism of a group.

(c) Find $U(9) \oplus U(12)$.

(d) Let $G = \{e, a, a^2, a^3\}$ be a cyclic group and $H = \{e, a^2\}$, where e is the identity of G. Show that H is a characteristic subgroup of G.

(e) Let $\mathbb{C}$ be the set of all complex numbers. Define a map $\phi : \mathbb{C} \to \mathbb{C}$ by $\phi(a + ib) = a - ib$, $a, b \in \mathbb{R}$. Show that ϕ is an automorphism.

(f) Define the set $U_k(n)$. Hence, find $U_7(105)$.

(g) Let G be a group. Define a map

$\phi : G \to G$ by $\phi(g) = g^{-1}$ for all $g \in G$ such that ϕ is an automorphism. Then show that G is abelian.

3. Answer the following questions : ***(any six)*** 5×6=30

(i) Prove that the commutator subgroup G' of a group G is nromal.

(ii) Let G be a finite group and $a \in G$ then show that $|cl(a)| = |G : C(a)|$, where $cl(a)$ is the conjugate class of a.

(iii) Prove that $\frac{G}{Z(4)} \cong I_{nn}(4)$,where $I_{nn}(4)$ is the set of all inner automorphism of G and $Z(4)$ is the centre of G.

(iv) Show that a characteristic subgroup of a group G is a normal subgroup of G. Is the converse true? Justify.

(v) If H is a Sylow p-subgroup of a finite group G and $x \in G$, then show that $x^{-1}Hx$ that is also a Sylow p-subgroup of G.

(vi) Find all conjugate classes in and S_4

verify the class equation.

(vii) Let G be a group acting on a set A under $*$. Let $a \in A$ be any fixed element. Then define stabilizer G_a in G. Show that G_a is a subgroup of G.

(viii) Let G and H be two finite cyclic groups. Prove that $G \oplus H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime.

(ix) Let $G_1, G_2, \ldots, G_n$ be a finite collection of groups. Prove that $G_1 \oplus G_2 \oplus \ldots \oplus G_n$ is a group under component-wise operation.

(x) Prove that any two Sylow p-subgroups of a finite group G are conjutage in G.

4. Answer the following questions: ***(any two)*** 2×10=20

(a) State and prove Generalized Cayley theorem. 2+8=10

(b) If G is a finite group and $P / |G|$ where P is a Prime number, then show that there exists an element $e \neq a \in G$ such that $|a| = p$.

(c) (i) Let G be a group and g be an element of G. Define a mapping $T_g : G \to G$ such that $T_g(x) = xgx^{-1} \forall x \in G$, show that is an automorphism of G.

(ii) Let G be a group and $a \in G$ then prove that $cl(a) = \{a\} \Leftrightarrow a \in Z(4)$ where $Z(4)$ is the centre of G. 5+5=10

(d) State and prove Index Theorem. 2+8=10

5. Answer the following questions: ***(any one)*** 14×1=14

(a) (i) Show that the alternating group A_5 is simple.

(ii) Suppose G is a group of finite order and p is a prime number. If $p^m \mid\mid G$ and $p^{m+1} \times \mid G \mid$, then prove that G has a subgroup of order p^m. 8+6=14

(b) (i) Determine the number of cyclic subgroups of order 10 in $\mathbb{Z}_{100} \oplus \mathbb{Z}_{25}$.

(ii) Show that every group of order 56 hs a proper non-trivial normal subgroup. 6+8=14

(c) (i) Let G be a group of order 15. Then show that either Sylow 3-subgroup or Sylow 5-subgroup is normal in G.

(ii) Let G be a group acting on a non-empgy set A under the group action $*$. For each fixed $g \in G$ define a mapping $\sigma_g : A \to A$ by $\sigma_g(a) = g * a$. Show that σ_g is a permutation on A.

(iii) Let G be a group of order p^2, where p is a prime. Then show that G is abelian. 4+5+5=14
