Total number of printed pages–11

63/1 (SEM–5) CC11/MATHC5116

2024

MATHEMATICS

Paper : MATHC 5116

(Multivariate Calculus)

Full Marks : 80

Pass Marks : 32

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Choose the correct answer : ***(any six)*** 1×6=6

   (a) The Domain and Range of the function $f(x, y) = \sqrt{y - x^2}$ is

   (i) Domain of f is $y \le x^2$ and Range of f is $[0, \infty]$

   (ii) Domain of f is $y \ge x^2$ and Range of f is $[0, \infty)$

Contd.

(iii) Domain of f is $y \geq x^2$ and Range of f is $(0,\infty)$

(iv) Domain of f is $y \leq x^2$ and Range of f is $(0, \infty)$

(b) The necessary condition for the function $f(x, y)$ to have an extreme value at (a, b) is

(i) $f_{xx}(a, b) = 0$, $f_y(a, b) = 0$

(ii) $f_{xx}(a, b) = 0$, $f_{yy}(a, b) = 0$

(iii) $f_x(a, b) = 0$, $f_y(a, b) = 0$

(iv) $f_{xy}(a, b) = 0$, $f_{yy}(a, b) = 0$

(c) The function

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2} & , (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0,0) \end{cases}$$ is

(i) Continuous at every point
(ii) Discontinuous at every point
(iii) Continuous at the origin
(iv) Continuous at every point except the origin

(d) The value of $\int_0^a \int_0^b \left(x^2 + y^2\right) dxdy$ is

(i) $\frac{1}{3}\left(a^2 + b^2\right)$

(ii) $\frac{1}{3}ab\left(a^2 + b^2\right)$

(iii) $\frac{1}{3}\left(a^3 + b^3\right)$

(iv) $\frac{1}{3}a^2b^2(a + b)$

(e) A point at which the function is neither maximum nor minimum is called

(i) Vector point function
(ii) Saddle point
(iii) Neighbourhood
(iv) Stationary point

(f) A vector f is said to be irrotational, if

(i) $\nabla \times f = 0$

(ii) $\nabla \times f \neq 0$

(iii) $\nabla . f = 0$

(iv) $\nabla .(\nabla \times f) = 0$

(g) If $f(x, y) = x\cos y + ye^x$, the value of $\frac{\partial^2 f}{\partial y \partial x}$ is

(i) $\cos y + e^x$

(ii) $\cos y + ye^x$

(iii) $\sin y + ye^x$

(iv) $-\sin y + e^x$

(h) The plane tangent to the surface $z = x\cos y - ye^x$ at (0, 0, 0) is

(i) $x + y + z = 0$

(ii) $x + y - z = 0$

(iii) $x - y - z = 0$

(iv) $x - y + z = 0$

(i) If $\vec{F} = 2x\hat{i} + 3y\hat{j} + 4z\hat{k}$, the value of $\nabla.\vec{F}$ is

(i) 0

(ii) −2

(iii) 8

(iv) 9

(j) The range of the function $f(x, y, z) = xy\log z$ is

(i) $(0, \infty)$

(ii) $[0, \infty)$

(iii) $(-\infty, \infty)$

(iv) $[-\infty, \infty]$

2. Answer the following questions : ***(any five)*** 2×5=10

(a) If $z = e^{xy^2}$, $x = t\cos t$, $y = t\sin t$, compute $\frac{dz}{dt}$ at $t = \frac{\pi}{2}$.

(b) Find the equation of the tangent plane and normal line to the surface $x^2 + 2y^2 + 3z^2 = 12$ at $(1, 2, -1)$.

(c) Evaluate

$\iint_R xy\,dxdy$, where R is the quadrant of the circle $x^2 + y^2 = a^2$, where $x \geq 0$ and $y \geq 0$.

(d) Find the local extreme value of the function

$$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4$$

(e) Prove that

$$\left(y^2 - z^2 + 3yz - 2x\right)\hat{i} + (3xz + 2xy)\hat{j}$$

$+(3xy - 2xz + 2z)\hat{k}$ is irrotational.

(f) Find the directional derivative of

$f(x, y) = 2x^2 - xy + 5$ at $(1, 1)$ in the

direction of unit vector $\bar{u} = \frac{1}{5}\left(3\hat{i} - 4\hat{j}\right)$.

(g) If a force $\vec{F} = 2x^2y\hat{i} + 3xy\hat{j}$ displaces a particle in the xy-plane from $(0, 0)$ to $(1, 4)$ along a curve $y = 4x^2$, find the work done.

3. Answer the following questions : *(any six)* 5×6=30

(a) Show that the function

$f(x, y) = \dfrac{2x^2y}{x^4 + y^2}$ has no limit as (x, y) approaches $(0, 0)$.

(b) Electrical resistors are in parallel. If registors of R_1, R_2 and R_3 ohms are connected in parallel to make an R-ohm resistor, the value of R can be found from the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Find the value $\dfrac{\partial R}{\partial R_2}$, when $R_1 = 30$, $R_2 = 45$ and $R_3 = 90$ ohms.

(c) Evaluate : $\displaystyle\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dxdy}{1 + x^2 + y^2}$

(d) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$, show that

$$div\left(\frac{\vec{r}}{|\vec{r}|^3}\right) = 0.$$

(e) Evaluate

$\displaystyle\iiint_R \left(x^2 + y^2 + z^2\right) dxdydz$ where R denotes the region bounded by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = a$, $a > 0$.

(f) Apply Green's theorem to evaluate $\int_C \left[\left(2x^2 - y^2\right) dx + \left(x^2 + y^2\right) dy\right]$, where C is the boundary of the area enclosed by the x-axis and the upper half of circle $x^2 + y^2 = a^2$.

(g) State and prove fundamental theorem for line integrals.

(h) Show that the function

$f(x, y) = x^3 + y^3 - 63(x + y) + 12xy$ is maximum at $(-7, 7)$ and minimum at $(3, 3)$.

(i) Find the surface area of the sphere $x^2 + y^2 + z^2 = 9$ lying inside the cylinder $x^2 + y^2 = 3y$.

(j) Let $\vec{F}$ be a constant vector field. Show that $div\ \vec{F} = 0$ and $curl\ \vec{F} = 0$.

4. Answer the following : ***(any two)*** 10×2=20

(a) (i) Show that the plane $3x + 12y - 6z - 17 = 0$ touches the conicoid $3x^2 - 6y^2 + 9z^2 + 17 = 0$. Find also the point of contact. 5

(ii) Evaluate

$$\int_0^3 \int_0^2 \left(4 - y^2\right) dy\, dx$$ 5

(b) (i) Using Lagrange multipliers method, find the stationary values of the function $f(x, y, z) = x^2 + y^2 + z^2$, given that $z^2 = xy + 1$. 5

(ii) Determine the constants a and b such that the curl of vector

$\vec{F} = (2xy + 3yz)\hat{i} + \left(x^2 + axz - 4z^2\right)\hat{j}$

$-(3xy + byz)\hat{k}$ is zero. 5

(c) (i) If $z = f(x, y)$ where $x = e^u \cos v$ and $y = e^u \sin v$, show that

$$y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = e^{2u}\frac{\partial z}{\partial y}.$$ 5

(ii) Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$. 5

(d) (i) Transform the integral to Cartesian form and hence evaluate : 5

$$\int_0^{\pi}\int_0^{a} r^3 \sin\theta\cos\theta \, dr d\theta$$

(ii) A rectangular box, open at the top, is to have a volume of 32 *cc*. Find the dimensions of the box requiring least material for its construction. 5

5. Answer ***any one*** of the following questions : 14×1=14

(a) (i) State and prove Stokes' theorem. 2+8=10

(ii) Use Stokes' theorem to find the line integral $\int_{\Gamma}(xdx + xydy + xyzdz)$, where Γ is the piece of the twisted cubic at $x = t$, $y = t^2$, $z = t^3$ corresponding to the interval $0 \le t \le 1$. 4

(b) (i) Verify Gauss divergence theorem for the surface integral

$$\iint_S \left(x^2 - yz\right) dydz + \left(y^2 - zx\right) dzdx + \left(z^2 - xy\right) dxdy$$

where S is the surface bounded by the rectangular parallelepiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$. 10

(ii) Determine the area of the ellipse given below using line integral,

$$\frac{x^2}{16} + \frac{y^2}{4} = 1.$$ 4

(c) (i) A necessary and sufficient condition that the vector field defined by the vector point function $\vec{F}$ with continuous derivatives is an open region R be conservative is that $\nabla \times \vec{F} = \vec{0}$, where the operator

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}.$$ 7

(ii) Evaluate $\iint_S xdS$, where S is the entire surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$, $z = x + 2$. 7