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63/1 (SEM–6) CC13/MATHC 6136

2024

MATHEMATICS

Paper : MATHC 6136

(Metric Spaces and Complex Analysis)

Full Marks : 80

Pass Marks : 32

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Choose the correct answer ***(any six)*** : $1\times6=6$

   (a) $\underset{z\to -1}{Lt}\ \dfrac{iz+3}{z+1}$ is equal to

   (i) 3

   (ii) 0

   (iii) ∞

   (iv) 1

Contd.

(b) In the metric space (R, d), where d is the usual metric on R, if $A = (0, 1]$ then

(i) $\bar{A} = (0, 1]$

(ii) $\text{int}(A) = (0, 1]$

(iii) $\text{int}(A) = (0, 1)$

(iv) $D(A) = (0, 1]$

(c) If the function $f(z) = u(x, y) + iv(x, y)$ is analytic at any point $z = x + iy$, then

(i) $u_x = v_x, \quad u_y = v_y$

(ii) $u_x = v_y, \quad u_y = -v_x$

(iii) $u_x = -v_y, \quad u_y = v_x$

(iv) $u_x = -v_x, \quad u_y = v_y$

(d) Which of the following is not true in a metric space?

(i) Finite union of open sets are open.

(ii) Finite intersection of open sets are open.

(iii) Arbitrary union of open sets are open.

(iv) Arbitrary intersection of open sets are open.

(e) Let $f(z)$ be continuous on a contour L of length l and $|f(z)| \le M$ on L. Then

(i) $\int_L |f(z)| dz \le ML$

(ii) $|\int_L f(z) dz| \le ML$

(iii) $\int_L f(z) |dz| \le ML$

(iv) $\int_L f(z) dz \le ML$

(f) Let (X, d) be a complete metric space and Y be a subset of X. Then which of the following is true?

(i) (Y, d) is complete if Y is open.

(ii) (Y,d) is complete if Y is closed.

(iii) If Y is closed then (Y,d) is not complete.

(iv) If Y is open then (Y,d) is complete.

(g) If ϕ satisfies the differential equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0,$$ then ϕ is called

(i) Cauchy's function

(ii) Cauchy-Riemaan functions

(iii) analytic function

(iv) harmonic function

(h) Let (X,d_1) and (Y,d_2) be two metric spaces. Then a function $f: X \to Y$ is homomorphism, if

(i) f is bijective and f is continuous

(ii) f is bijective and f, f^{-1} both are continuous

(iii) f is one-one and f, f^{-1} both are continuous

(iv) f is onto and f, f^{-1} both are continuous

(i) If $\underset{z\to z_0}{Lt} f(z)$ exists, then which of the following is not necessarily correct value of the limit ?

(i) α, if $\underset{z\to z_0}{Lt} f(z)=\alpha$, where the point z approaches z_0 along a line through the point $z=z_0$

(ii) β, if $\underset{z\to z_0}{Lt} f(z)=\beta$, where the point z approaches z_0 along a circle passing through the point $z=z_0$.

(iii) $f(z_0)$

(iv) Unique

(j) In the metric space (R, d), where d is usual metric, the diameter of the set $\{0, 1, 2, 3, \cdots, 100\}$ is

(i) 100

(ii) 0

(iii) 101

(iv) 99

2. Answer the following questions : ***(any five)*** $2 \times 5 = 10$

(a) Using definition of exponential function of $\sin z$ and $\cos z$. Show that $\sin^2 z + \cos^2 z = 1$.

(b) In a metric space (X, d), prove that the intersection of finite number of open sets is open.

(c) Evaluate : $\int_0^{\pi/4} e^{it} dt$

(d) Prove that every convergent sequence in a metric space is a Cauchy sequence.

(e) Evaluate $f'(z)$ from the definition where $f(z) = 1/z$.

(f) Define uniform continuity on metric space. Is uniform continuous function on metric space always continuous ?

(g) Prove that $\underset{z \to 0}{Lt} \frac{\bar{z}^2}{z} = 0$

3. Answer the following questions : ***(any six)*** $5 \times 6 = 30$

(a) Find an analytic function $f(z)$ whose real part is $e^x \cos y$.

(b) Let the mapping $d : R \times R \to R$ be defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Show that (R, d) is a metric space.

(c) Evaluate : $\underset{z \to 0}{Lt} (\cos z)^{\frac{1}{z^2}}$

(d) In a metric space, show that every closed sphere is a closed set.

(e) Solve : $e^z = -2$

(f) Let (X, d) and (Y, ρ) be metric spaces. Prove that a function $f : X \to Y$ is continuous at $x_0 \in X$ if and only if to every sequence $\langle x_n \rangle$ in X converges to x_0 implies the sequence $\langle f(x_n) \rangle$ in Y converges to $f(x_0)$.

(g) Evaluate :

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$$

where C is the circle $|z| = 3$.

(h) Define contraction mapping. Let d be a usual metric for R^2 and the mapping $f : R^2 \to R^2$ defined by $f(x) = x/2$, $\forall x \in R^2$ where $x = (x_1, x_2)$. Show that f is a contraction on R^2.

(i) State and prove Cauchy-Goursat theorem for multiply-connected regions.

(j) Evaluate $\int_C \frac{dz}{z}$, where C is the top-half of the circle $|z| = 1$ from $z = 1$ to $z = -1$.

4. Answer the following questions : ***(any two)*** 10×2=20

(a) State and prove Cauchy-Riemann equations of an analytic function in polar form. 2+8=10

(b) Let A be a subset of a metric space X. Show that the following statements are equivalent :

(i) A is non-dense in X.

(ii) $\bar{A}$ contains no-neighbourhood.

(iii) $(\bar{A})'$ is dense in X.

(c) If f be analytic everywhere inside and on a simply closed contour C, taken in the positive sense and z_0 is any point interior to C, then prove that

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$$

(d) Prove that the metric space (R, d) is complete, where d denotes the usual metric for R.

5. Answer the following questions : ***(any one)*** 14

(a) (i) Let (X, d) be a complete metric space and let f be a contraction mapping on X. Then prove that there exists one and only one point x in X such that $f(x) = x$. 10

(ii) Define interior of a set and limit point of a set. 2+2=4

(b) (i) State and prove Taylor's theorem.

(ii) Show that $\sum_{n=1}^{\infty} \frac{z^n}{1+z^{2n}}$ are uniformly convergent in $|z| < 1$.

2+8+4=14

(c) Let $c[a, b]$ denote the set of all continuous functions on $[a, b]$. For $f, g \in C[a, b]$, define

$$\rho(f, g) = \left\{ \int_a^b |f(x) - g(x)|^2 \, dx \right\}^{1/2}$$

Show that ρ is a metric on $c[a, b]$. Also show that this metric is not complete. 6+8=14