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2024

MATHEMATICS

Paper : MATHC6146

(Ring Theory and Linear Algebra–II)

Full Marks : 80

Pass Marks : 32

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Choose the correct answer : ***(any six)*** $1 \times 6 = 6$

   (i) Let F be a field and $f(x) \in F[x]$. Let $a \in F$ be a zero of multiplicity 3. Then

   (A) Both $(x-a)^3$ and $(x-a)^4$ are factors of $f(x)$

(B) $(x-a)^4$ is a factor of $f(x)$ but $(x-a)^3$ is not a factor of $f(x)$

(C) $(x-a)^3$ is a factor of $f(x)$ but $(x-a)^4$ is not a factor of $f(x)$

(D) Neither $(x-a)^3$ nor $(x-a)^4$ is a factor of $f(x)$

(ii) Which one of the following is correct, where ED : Euclidean Domain, PID : Principal Ideal Domain, UFD : Unique Factorization Domain ?

(A) $ED \Rightarrow PID \Rightarrow UFD$

(B) $UFD \Rightarrow PID \Rightarrow ED$

(C) $ED \Rightarrow PID \not\Rightarrow UFD$

(D) $ED \not\Rightarrow PID \Rightarrow UFD$

(iii) Let $P = \langle x^2+1 \rangle$ be an ideal in $\mathbb{Z}[x]$. Then which of the following is correct ?

(A) P is both prime and maximal

(B) P is maximal but not prime

(C) P is prime but not maximal

(D) P is neither prime nor maximal

(iv) Let $f(x) = 3x^5 + 15x^4 - 20x^3 + 10x + 20 \in \mathbb{Z}[x]$. Then

(A) $f(x)$ is reducible over $\mathbb{Q}$ but irreducible over $\mathbb{Z}$

(B) $f(x)$ is reducible over both $\mathbb{Q}$ and $\mathbb{Z}$

(C) $f(x)$ is irreducible over $\mathbb{Q}$

(D) $f(x)$ is irreducible over $\mathbb{Z}_5$

(v) Let V^* be the dual space of a finite dimensional vector space V and V^{**} be the double dual of V. Then

(A) $V \cong V^*$ but $V \ncong V^{**}$

(B) $V \cong V^{**}$ but $V \ncong V^*$

(C) $V \ncong V^*$ and $V \ncong V^{**}$

(D) $V \cong V^*$ and $V \cong V^{**}$

(vi) Which of the following statements is false in the case of a finite dimensional vector space V ? (V^* denotes the dual of V)

(A) Every linear transformation on V is a linear functional.

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(B) Every linear functional is a linear transformation on the vector space V.

(C) If V is isomorphic to a finite dimensional vector space W, then $V^* \cong W^*$.

(D) $dim V = dim V^*$

(vii) Let T be a linear transformation on a finite dimensional vector space V such that $dim V = n$. Let $\{\lambda_1, \lambda_2, \ldots\ldots, \lambda_n\}$ be the distinct eigenvalues of T corresponding to the ordered basis $\beta = \{v_1, v_2, \ldots\ldots, v_n\}$, where v_i are eigenvectors of λ_i for each i. Then which of the following statements is false ? [$Tr(T)$ denotes trace of T]

(A) $Tr\left([T]_\beta\right) = \sum_{i=1}^{n} \lambda_i$

(B) $[T]_\beta$ is diagonalisable whose diagonal elements are $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$

(C) $Tr\left([T]_\beta\right) \neq \sum_{i=1}^{n} \lambda_i$

(D) $v_i \in N(T - \lambda_i I)$ for each i, where $N(T - \lambda_i I)$ denotes the null space of $T - \lambda_i I$.

(viii) Let T be any linear operator on a finite dimensional vector space V. Then which of the following statements is not correct ?

(A) For every eigenvalue λ of T, the eigenspace E_λ is T-invariant.

(B) The range space $R(T)$ is T-invariant.

(C) The null space $N(T)$ is T-invariant.

(D) T need not necessarily to have an invariant subspace always.

(ix) Let $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$, $v_1 = (1, -1)$ and $v_2 = (1, 2)$. Then

(A) v_1 is an eigenvector of A but v_2 is not an eigenvector of A

(B) v_2 is an eigenvector of A but v_1 is not an eigenvector of A

(C) Both v_1 and v_2 are eigenvectors of A

(D) Neither v_1 nor v_2 is an eigenvectors of A

(x) Let $\langle .,. \rangle$ be an inner product on a vector space X. Then which of the following statements is correct ?

(A) $\langle .,. \rangle$ is linear in both the components.

(B) $\langle .,. \rangle$ is linear in the first component.

(C) $\langle .,. \rangle$ is linear in the second component.

(D) $\langle .,. \rangle$ is not linear in both the components.

2. Answer the following questions : ***(any five)*** $2 \times 5 = 10$

(i) Let $f(x) = 2x^3 + x^2 + 2x + 2$ and $g(x) = 2x^2 + 2x + 1$ in $\mathbb{Z}_3[x]$. Then find $f(x) + g(x)$ and $f(x) \cdot g(x)$.

(ii) Let $f(x) = x^3 + x^2 + x + 1 \in \mathbb{Z}_2[x]$. Write $f(x)$ as a product of irreducible polynomials over $\mathbb{Z}_2$.

(iii) For the following functions f on a vector space $V(F)$, determine which one of them is a linear functional and why.

(A) $V = \mathbb{R}^2, f(x, y) = (2x, 4y)$

(B) $V = M_{2\times 2}(F), f(A) = Tr(A)$, where $Tr(A)$ denotes the trace of A.

(iv) Let $\beta = \{(2, 1), (3, 1)\}$ be an ordered basis for $\mathbb{R}^2$. Suppose that $\beta^* = \{f_1, f_2\}$ defined by $f_1(x, y) = -x + 3y$ and $f_2(x, y) = x - 2y$. Show that β^* is a dual basis for $(\mathbb{R}^2)^*$.

(v) Let V be an inner product space and suppose that x and y are orthogonal. Then prove that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

(vi) Let T be a linear operator on a complex inner product space V with an adjoint T^*. Then, show that if T is self-adjoint, $\langle T(x), x\rangle$ is real for all $x \in V$.

(vii) Let $x = (2, 1+i, i)$, $y = (2-i, 2, 1+2i)$ in $\mathbb{C}^3$. Then find $\|x+y\|$.

3. Answer the following questions : ***(any six)*** 5×6=30

(i) Define a primitive polynomial in $\mathbb{Z}[x]$. Prove that the product of two primitive polynomials is primitive. 1+4=5

(ii) Let F be a field and $p(x)$ be an irreducible polynomial over F. Then prove that $\frac{F[x]}{\langle p(x)\rangle}$ is a field.

(iii) Prove that in a principal ideal domain, an element is an irreducible if and only if it is a prime.

(iv) Define a minimal polynomial of a linear operator on a vector space. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $T(a, b) = (2a+5b, 6a+b)$. Then find the minimal polynomial of T with respect to the standard ordered basis of $\mathbb{R}^2$. 1+4=5

(v) Let $V(F)$ be a finite dimensional vector space with dual space V^*. For every $x \in V$, define $\hat{x} : V^* \to F$ by $\hat{x}(f) = f(x)$ for every $f \in V^*$. Then prove that $\hat{x} \in V^{**}$. Also, prove that if $\hat{x}(f) = 0$ for every $f \in V^*$, then $x = 0$. 2+3=5

(vi) Verify Cayley-Hamilton Theorem for the linear transformation T defined on $\mathbb{R}^2$ by $T(a, b) = (a+3b, a+b)$.

(vii) Let $x=(a_1,a_2,a_3), y=(b_1,b_2,b_3)\in\mathbb{C}^3$. Define $\langle x,y\rangle=\sum_{i=1}^{3}a_i\overline{b_i}$. Show that $\langle .,.\rangle$ is an inner product on $\mathbb{C}^3$.

(viii) Define a norm on a vector space. Let $\|\cdot\|$ be a norm on vector space V, and define for each ordered pair of vectors, the scalar $d(x,y)=\|x-y\|$. Prove that d is a metric on V. 1+4=5

(ix) Let V be an inner product space over a field F. Let T, U be linear operators on V. Then prove the following :
(T^* denotes the adjoint of T)

(a) $(T+U)^*=T^*+U^*$

(b) $(cT)^*=\bar{c}T^*$ for any $c\in F$.

2½×2=5

(x) Find the minimal solution of the following equations :

$x+2y+z=4,\ x-y+2z=-11,$
$x+5y=19$

4. Answer the following questions : ***(any two)*** 10×2=20

(i) Let F be a field and $f(x),\ g(x)\in F[x]$ with $g(x)\neq 0$. Then prove that $f(x)=g(x)q(x)+r(x)$ for some uniquely determined polynomials $q(x)$ and $r(x)$ in $F[x]$ with the condition that either $r(x)=0$ or $\deg r(x)<\deg g(x)$. Hence, for the polynomials $f(x)=3x^4+x^3+2x^2+1$ and $g(x)=x^2+4x+2$ in $\mathbb{Z}_5[x]$, find polynomials $q(x)$ and $r(x)$ in $\mathbb{Z}_5[x]$ such that $f(x)=g(x)q(x)+r(x)$ and verify it. 6+4=10

(ii) Let $A=\begin{pmatrix}4&0&1\\2&3&2\\1&0&4\end{pmatrix}$. Find the eigenvalues of A and then find their corresponding eigenspaces. Hence, show that the matrix A is diagonalisable and find the diagonal matrix of it. 2+5+2+1=10

Contd.

(iii) (a) Let T be a self-adjoint operator on a finite dimensional inner product space V. Then prove that every eigenvalue of T is real. 2

(b) Let V be a complex inner product space, and let T be a linear operator on V with its adjoint T^*. Define $T_1 = \frac{1}{2}(T + T^*)$ and $T_2 = \frac{1}{2i}(T - T^*)$. Then prove that T_1 and T_2 are self-adjoint and $T = T_1 + iT_2$. Moreover, show that T is normal if and only if $T_1T_2 = T_2T_1$. 4+4=8

(iv) Let V be an inner product space and let T be a normal operator on V. Then prove that

(a) $\|T(x)\| = \|T^*(x)\|$ for all $x \in V$.

(b) $T - cI$ is normal for every scalar c.

(c) If x is an eigenvector of T, then x is also an eigenvector of T^*.

(d) If λ_1, λ_2 are distinct eigenvalues of T with corresponding eigenvectors x_1 and x_2, then x_1 and x_2 are orthogonal.

2½×4=10

5. Answer the following questions : ***(any one)***

14×1=14

(i) Let V be an inner product space and let $S = \{w_1, w_2, \ldots\ldots, w_n\}$ be a linearly independent subset of V. Define $S' = \{v_1, v_2, \ldots\ldots, v_n\}$, where $v_1 = w_1$ and

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j \text{ for } 2 \le k \le n.$$

Then prove that S' is an orthogonal set of nonzero vectors such that $span(S') = span(S)$. Hence, find an orthonormal set of the set $S = \{w_1, w_2, w_3\}$, where $w_1 = (1, 0, 1, 0)$, $w_2 = (1, 1, 1, 1)$, $w_3 = (0, 1, 2, 1)$.

8+6=14

(ii) (a) Let F be a field. If $f(x) \in F[x]$ and $deg\, f(x)$ is 2 or 3, then prove that $f(x)$ is reducible over F if and only if $f(x)$ has a zero in F. Hence, discuss the reducibility and irreducibility of the polynomial x^2+1 over $\mathbb{Z}_3$ and $\mathbb{Z}_5$. 6+3=9

(b) Find all zeros and their multiplicities of the polynomial $x^5+4x^4+4x^3-x^2-4x+1$ over $\mathbb{Z}_5$. 5

(iii) (a) Let T be a linear transformation on a vector space V and let $\lambda_1, \lambda_2, \ldots\ldots, \lambda_k$ be distinct eigenvalues of T. If $v_1, v_2, \ldots\ldots, v_k$ are eigenvectors of T corresponding to $\lambda_i (1 \le i \le k)$, then prove that $\{v_1, v_2, \ldots\ldots, v_k\}$ is linearly independent. 7

(b) Let T be a linear transofrmation on $\mathbb{R}^4$ defined by $T(a, b, c) = (2a-b, a+b, c-d, c+d)$, and let $W_1 = \{(s, t, 0, 0) \mid s, t \in \mathbb{R}\}$ and $W_2 = \{(0, 0, s, t) \mid s, t \in \mathbb{R}\}$. Show that W_1 and W_2 are T-invariant subspaces of $\mathbb{R}^4$. Also, show that $\mathbb{R}^4 = W_1 \oplus W_2$. 2+2+3=7

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