63/1 (SEM–5) DSE2/MATHE5026

2022

(Held in 2023)

MATHEMATICS

Paper : MATHE5026

(Probability and Statistics)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Choose the correct answer of the following : $1\times6=6$

(a) The probability of an impossible event is

(i) 0

(ii) 1

(iii) 2

(iv) 3

(b) If the events A and B are independent, then

(i) $P(A+B)=P(A)+P(B)$

(ii) $P(AB)=P(A)P(B)$

(iii) $P(AB)=P(A)$

(iv) $P(AB)=P(B)$

(c) Which of the following is a continuous distribution?

(i) Binomial distribution

(ii) Poisson distribution

(iii) Geometric distribution

(iv) Exponential distribution

(d) The binomial distribution have number of parameters

(i) one

(ii) two

(iii) three

(iv) four

(e) The correlation coefficient $\rho(X, Y)$ between X and Y is

(i) $\rho(X, Y) = \dfrac{\text{cov}(X, Y)}{\sigma_X \sigma_Y}$

(ii) $\rho(X, Y) = \dfrac{\text{cov}(X, Y)}{\sigma_X^2}$

(iii) $\rho(X, Y) = \dfrac{\text{cov}(X, Y)}{\sigma_Y^2}$

(iv) None of the above

(f) Let $f(x, y)$ be the joint probability density function of two independent random variables X and Y. Then

(i) $f(x, y) = 1, \forall (x, y)$

(ii) $f(x, y) = f_X(x) \cdot f_Y(y)$

(iii) $f(x, y) = f_X(x)/f_Y(y)$

(iv) $f(x, y) = f_Y(y)/f_X(x)$

2. Answer the following questions : 2×5=10

(a) Define axiomatic definition of probability.

(b) If A and B are events in a sample space S and $A \subset B$, then prove that $P(A) \leq P(B)$.

(c) If A and B are two independent events in a sample space S, then prove that A and $\overline{B}$ are also independent.

(d) Find $E(X)$ when the probability density function of X is

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(e) State weak law of large numbers.

3. Answer any *six* of the following questions : 5×6=30

(a) If A, B and C are any three events in a sample space S, then prove that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$ 5

(b) Let X be a continuous random variable whose probability density function is given by

$$f(x) = k(4x - 2x^2) \; ; \quad 0 < x < 2$$
$$= 0 \; ; \quad \text{elsewhere}$$

(i) Find k.

(ii) Find $P(X > 1)$. 2½+2½=5

(c) Explain Markov chain. 5

(d) Derive moment generating function of normal distribution. 5

(e) For any random variable X, prove that $\text{var}(aX + b) = a^2 \text{var}(X)$, where a, b are real constant. 5

(f) Derive the characteristic function of binomial distribution. 5

(g) The joint probability density function of X and Y is defined as follows

$$f(x, y) = Cx, \; 0 < y < x < 1, \text{ where } C \text{ is a constant}$$
$$= 0, \quad \text{elsewhere}$$

Find—

(i) the constant C;

(ii) the marginal density functions of X and Y. 1+4=5

(h) The joint probability density function of X and Y is given by

$$f(x, y) = \frac{2}{3}(x + 2y) \text{ for } 0 < x < 1, \; 0 < y < 1$$
$$= 0, \quad \text{elsewhere}$$

Find the conditional mean and conditional variance of X given $Y = \frac{1}{2}$. 2½+2½=5

(i) The probability density function of a random variable X is given by

$$f(x) = 2(1 - x) \text{ for } 0 < x < 1$$
$$= 0, \quad \text{elsewhere}$$

(i) Show that

$$E(X^r) = \frac{2}{(r+1)(r+2)}$$

(ii) Use this result to evaluate $E[(2X+1)^2]$.

2½+2½=5

4. Answer any *two* of the following questions : 10×2=20

(a) State and prove central limit theorem for independent and identically distributed random variables with finite variance. 2+8=10

(b) (i) State and prove Bayes' theorem. 1+4=5

(ii) A laboratory blood test is 95 percent effective in detecting a certain disease when it is, in fact present. However, the test also yields a 'false positive' result for 1 percent of the healthy persons test. If 0·5 percent of the population actually has the disease, what is the probability a person has the disease given that his test result is positive? 5

(c) (i) Define mean, moments, central moments, variance and standard deviation of a random variable X.

1+1+1+1+1=5

(ii) If the random variables X, Y and Z have the means $\mu_X = 2$, $\mu_Y = -3$, $\mu_Z = 4$ and variances $\sigma_X^2 = 1$, $\sigma_Y^2 = 5$, $\sigma_Z^2 = 2$ and covariances $\text{cov}(X, Y) = -2$, $\text{cov}(X, Z) = -1$, $\text{cov}(Y, Z) = 1$, then find mean and variance of $W = 3X - Y + 2Z$.

2½+2½=5

5. Answer any *one* of the following questions : 14

(a) Derive Chapman-Kolmogorov forward and backward equations for the birth and death process. 7+7=14

(b) (i) Define distribution function or cumulative distribution of a discrete random variable X. Find the distribution function of the total number of heads obtained in four tosses of a balanced coin. 1+6=7

(ii) Define probability density function of a continuous random variable X. Find a probability density function for the random variable X whose distribution function is given by

$$f(x) = \begin{cases} 0 & \text{for } x \le 0 \\ x & \text{for } 0 < x < 1 \\ 1 & \text{for } x \ge 1 \end{cases}$$

2+5=7

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