63/1 (SEM–5) DSE1/MATHE5016

2022

(Held in 2023)

MATHEMATICS

Paper : MATHE5016

(Number Theory)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Choose the correct option from the following : $1\times6=6$

(a) Which of the following congruences has unique incongruent solution?

(i) $8x \equiv 14 \pmod{24}$

(ii) $20x \equiv 14 \pmod{15}$

(iii) $12x \equiv 14 \pmod{2}$

(iv) $15x \equiv 14 \pmod{8}$

(b) The unit digit of 2^{100} is

(i) 2 *(ii)* 4

(iii) 6 *(iv)* 8

(c) The number of positive divisors of 50000 is

(i) 20

(ii) 30

(iii) 300

(iv) 400

(d) If p is a prime, then $\tau(p)$ is

(i) 1

(ii) 2

(iii) 3

(iv) 4

(e) For $a, m \in Z$, $a^{\phi(m)} \equiv 1 \pmod{m}$ if

(i) $(a, m) \neq 1$

(ii) $(a, m) = 1$

(iii) $m | a$

(iv) None of the above

(f) The number of quadratic non-residues modulo 23 is

(i) 10

(ii) 22

(iii) 11

(iv) 2

2. Answer the following questions : 2×5=10

(a) If $7 \nmid a$, prove that either $a^3 + 1$ or $a^3 - 1$ is divisible.

(b) Find $\phi(\phi(1001))$, where ϕ is Euler's phi function.

(c) Find the highest power of 7 contained in 2000!.

(d) List the primitive roots of 6.

(e) Show that the order of 2 modulo 5 is 4.

3. Answer any *six* of the following questions : 5×6=30

(a) Find all the possible solutions in positive integers of the Diophantine equation

$$56x + 72y = 40$$

(b) Prove that every positive integer $n > 1$ can be expressed uniquely as a product of primes.

(c) Solve the linear congruence $6x \equiv 15 \pmod{21}$.

(d) Show that $\sigma(mn) = \sigma(m)\sigma(n)$. Hence find $\sigma(180)$.

(e) If F and f are two arithmetic functions and

$$F(n) = \sum_{d|n} f(d)$$

prove that

$$f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right)$$

(f) For any positive integer n, show that $\phi(n) = n-1$, if and only if n is a prime.

(g) If $m > 2$, $n > 2$ and $(m, n) = 1$, then prove that integer mn has no primitive roots.

(h) Let p is an odd prime and $\gcd(a, p) = 1$. Prove that $x^2 \equiv a \pmod{p^n}$; $n \geq 1$ has a solution iff $\left(\frac{a}{p}\right) = 1$.

(i) If p is a prime and

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \ a_n \not\equiv 0 \pmod{p}$$

is a polynomial of degree $n \geq 1$ with integral coefficient, show that $f(x) \equiv 0 \pmod{p}$ has at most n incongruent solution modulo p.

4. Answer any *two* of the following questions : 10×2=20

(a) *(i)* If $d = (a, n)$, prove that the linear congruence $ax \equiv b \pmod{n}$ has a solution if and only if $d|b$. 5

(ii) If (x_0, y_0) is any particular solution of $ax + by = c$, show that the general solutions are

$$x_1 = x_0 + \frac{b}{d}k, \ y = y_0 - \frac{a}{d}k$$

where $k \in Z$ and $d = \gcd(a, b)$. 5

(b) *(i)* If the integer $n > 1$ has the prime factorization $n = p_1^{k_1} \cdot p_2^{k_2} \cdots p_r^{k_r}$, then show that $\tau(n) = \prod_{i=1}^{r}(k_i + 1)$. Hence show that τ is a multiplicative function. 5

(ii) For each positive integer $n \geq 1$, show that

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1 \end{cases}$$ 5

(c) *(i)* If p and q are both distinct odd primes, then prove that

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)}$$ 5

(ii) Prove that the solutions of the Pythagorean equation $x^2 + y^2 = z^2$ satisfying the conditions

$$\gcd(x, y, z) = 1, \ 2 | x$$

and $x > 0$, $y > 0$, $z > 0$ are

$$x = 2mn$$
$$y = m^2 - n^2$$
$$z = m^2 + n^2$$

for integers $m, n \in N$ such that $(m, n) = 1$ and $m \not\equiv n \pmod 2$. 5

5. Answer any *one* of the following questions : 14

(a) *(i)* Find the smallest positive integer that gives remainders 5, 2, 31 when divided by 6, 11, 35 respectively. 5

(ii) State and prove Wilson theorem. 5

(iii) If p be an odd prime, show that congruence $x^2 \equiv -1 \pmod p$ has a solution if and only if $p \equiv 1 \pmod 4$. 4

(b) *(i)* Define Möbius μ function. Show that μ is a multiplicative function. 5

(ii) For each positive integer show that

$$n = \phi(d_1) + \phi(d_2) + \cdots + \phi(d_k)$$

where $d_1, d_2, \cdots, d_k$ are all divisors of n. 5

(iii) If n is the product of two prime numbers, show that

$$\phi(n)\sigma(n) = (n+1)(n-3)$$ 4

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