63/1 (SEM–5) CC12/MATHC5126

2 0 2 2

(Held in 2023)

MATHEMATICS

Paper : MATHC5126

(Group Theory—II)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Choose the correct answer of the following : $1\times6=6$

(a) Let G be a group. Then which one of the following is false?

(i) $|T(a)| = |a|$ for all $a \in G$ and for all $T \in \text{Aut}(G)$

(ii) $|bab^{-1}| = |a|$ for all $a, b \in G$

(iii) $|\text{Aut}(G)| = \infty$ if G is an infinite cyclic group

(iv) $|\text{Aut}(G)| = |u(n)|$ if G is a cyclic group of order n

(Turn Over)

(b) Let $\phi \in \text{Aut}(G)$, where $G = \mathbb{Z}_{50}$ and let $\phi(7) = 13$. Then which one of the following is true?

(i) $\phi(x) = 13x$ for all $x \in G$

(ii) $\phi(x) = 9x$ for all $x \in G$

(iii) $\phi(x) = 7x$ for all $x \in G$

(iv) $\phi(x) = x$ for all $x \in G$

(c) Let G be a finite group. Then which of the following is not a class equation?

(i) $|G| = \sum_{a \in G} |G : c(a)|$

(ii) $|G| = |Z(G)| + \sum_{a \notin Z(G)} |G : C(a)|$

(iii) $|G| = |Z(G)| + \sum_{a \notin Z(G)} |G : N(a)|$

(iv) None of the above

(d) Let G be a finite abelian group and let a be any element of G. Then

(i) $|\text{Cl}(a)| = 1$

(ii) $|\text{Cl}(a)| = |N(a)|$

(iii) $|\text{Cl}(a)| = |C(a)|$

(iv) $|\text{Cl}(a)| = |G|$

(Continued)

(e) Consider the elements (3, 7) and (7, 9) in $u(8) \oplus u(10)$. Then the product $(3, 7) \cdot (7, 9)$ is equal to

(i) (5, 3)

(ii) (21, 63)

(iii) (5, 7)

(iv) None of the above

(f) Let G and H be two finite cyclic groups. Then which of the following is true?

(i) $G \oplus H$ is always cyclic

(ii) $G \oplus H$ is cyclic if and only if $\gcd(|G|, |H|) \neq 1$

(iii) $G \oplus H$ is cyclic if and only if $\gcd(|G|, |H|) = 1$

(iv) None of the above

2. Answer the following questions : 2×5=10

(a) Find $(3, 5) + (6, 9)$ and $5(6, 9)$ in $\mathbb{Z}_8 \oplus \mathbb{Z}_{10}$. 1+1=2

(b) Find a commutator of (23) and (123) in S_3.

(c) Let $\sigma_1 = (1)(35)(89)(2476)$ and $\sigma_2 = (3)(47)(81)(5269)$ belong to S_9. Then find a τ in S_9 such that $\tau\sigma_1\tau^{-1} = \sigma_2$.

(Turn Over)

(d) Let the group G act on the nonempty set A. For each fixed $g \in G$, define a mapping $\sigma_g : A \to A$ by $\sigma_g(a) = g \circ a$. Show that σ_g is a permutation on A.

(e) Let k, n be positive integers such that k divides n. Then define $u_k(n)$. Hence find $u_7(105)$. 1+1=2

3. Answer any *six* of the following questions : 5×6=30

(a) Let G be a group. Prove that the mapping $\alpha(g) = g^{-1}$ for all $g \in G$ is an automorphism if and only if G is abelian.

(b) Define a characteristic subgroup of a group. Prove that the centre of a group G is a characteristic subgroup of G. 1+4=5

(c) Prove that the order of an element in an external direct product of a finite number of finite groups is the least common multiple of the orders of the components of the element. Hence, find $|(4, 2, 3)|$ in $\mathbb{Z}_8 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_4$. 4+1=5

(d) Let $G_1, G_2, \cdots, G_n$ be a finite collection of groups. Prove that $G_1 \oplus G_2 \oplus + \cdots \oplus G_n$ is a group under componentwise operation.

(Continued)

(e) Let G be a group acting on the nonempty set S and let s be a fixed element in S. Then define stabilizer G_s of s in G. Show that G_s is a subgroup of G. 1+4=5

(f) Let G be a group acting on the nonempty set A. Prove that the relation on A defined by $a \sim b$ if and only if $a = g \circ b$ for some $g \in G$ is an equivalence relation. Further, prove that for each $a \in A$, the number of elements in the equivalence class containing a is $|G : G_a|$, the index of the stabilizer G_a of a. 2½+2½=5

(g) Let G be a finite group and let a be an element of G. Prove that

$$|\mathrm{Cl}(a)| = |G : C(a)|$$

(h) Find the number of elements of order 5 in $\mathbb{Z}_{25} \oplus \mathbb{Z}_5$.

(i) Prove that the commutator subgroup G' of a group G is normal.

4. Answer any *two* of the following questions : 10×2=20

(a) (i) Define a Sylow p-subgroup of a finite group.

(Turn Over)

(ii) State Sylow's 3rd theorem of a finite group. Apply this theorem to show that a group of order 72 is not simple. 1+1+8=10

(b) *(i)* Let G be a group and let g be an element of G. Define a mapping

$$T_g : G \to G \text{ s.t. } T_g(x) = xgx^{-1} \ \forall \ x \in G$$

Show that T_g is an automorphism of G.

(ii) Let G be a group and $I_{nn}(G)$ is the inner automorphism group of it. Prove that

$$\frac{G}{Z(G)} \approx I_{nn}(G)$$

4+6=10

(c) Prove that an integer of the form $2 \cdot n$, where n is an odd number greater than 1, is not the order of a simple group. Justify the theorem by taking a group of order 42. 7+3=10

5. Answer any *one* of the following questions : 14

(a) *(i)* Let $G = \{1, 7, 17, 23, 49, 55, 65, 71\}$ be the group under multiplication modulo 96. Express G as an external and internal direct product of cyclic groups.

(ii) Let G be a group and let a be an element of G. Then prove that

$$\mathrm{Cl}(a) = \{a\} \Leftrightarrow a \in Z(G)$$

10+4=14

(b) *(i)* Let G be a finite group and let p be a prime that divides the order of G. Then prove that G has an element of order p.

(ii) Determine $\mathrm{Aut}(\mathbb{Z}_6)$ completely. 10+4=14

★★★