2022

(Held in 2023)

MATHEMATICS

Paper : MATHC5116

(Multivariate Calculus)

Full Marks : 80
Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following as directed : 1×6=6

(a) If $f(x, y, z) = x^2ye^{2x} + (x+y-z)^2$, then find $\frac{d}{dx}f(x, x, x)$.

(b) Find $\frac{\partial(x, y)}{\partial(r, \theta)}$, if $x = r\cos\theta$ and $y = r\sin\theta$.

(c) What is a Jordan curve?

(d) Find the domain of

$$f(x, y) = \sqrt{16 - x^2 - y^2}$$

(e) Evaluate $\int_0^1 \int_1^2 x^2 y^5 \, dxdy$.

(f) If $\vec{F} = x + 3y^2 - z^3$, $\vec{G} = 2x^2yz$, $\vec{H} = 2z^2 - xy$, then

$$\left.\frac{\partial(F, G, H)}{\partial(x, y, z)}\right|_{(1, -1, 0)}$$

is

(i) 8

(ii) 10

(iii) 12

(iv) None of the above

(Choose the correct answer)

2. Answer the following questions : 2×5=10

(a) If $f(x, y) = e^x \cos y$, then show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

(b) Find the slope of the line that is parallel to the *xz*-plane and tangent to the surface $z = x\sqrt{x+y}$ at the point $P(1, 3, 2)$.

(c) Find an equation in cylindrical coordinates for the elliptic paraboloid

$$z = x^2 + 3y^2$$

(Continued)

(d) If $\vec{F} = 2\hat{i} + 3\hat{j} + 4\hat{k}$, then find $\nabla\vec{F}$.

(e) Define surface integral.

3. Answer any *six* of the following questions : 5×6=30

(a) Let f be the function defined by

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

for $(x, y) \neq (0, 0)$.

(i) Find $\lim_{(x, y)\to(1, 2)} f(x, y)$.

(ii) Prove that f has no limit at $(0, 0)$. 2+3=5

(b) The temperature at a point (x, y) on a given metal plate in the *xy*-plane is determined according to the formula

$$T(x, y) = x^3 + 2xy^2 + y \text{ (in degrees)}$$

Compute the rate at which the temperature changes with distance, if we start at (2, 1) and move—

(i) parallel to the vector $\hat{i}$;

(ii) parallel to the vector $\hat{j}$. 2½+2½=5

(Turn Over)

(c) Find the directional derivative at

$$f(x, y) = e^{x^2 y^2}$$

at $P_0(1, -1)$ in the direction towards the vector $\vec{u} = 2\hat{i} + 3\hat{j}$.

(d) Evaluate the double integral by reversing the order

$$\int_0^1 \int_x^1 e^{y^2} dy dx$$

(e) Let $\vec{F}$ be a constant vector field. Show that div $\vec{F} = 0$ and curl $\vec{F} = 0$.

(f) Evaluate $\iiint_B z^2 y e^x dV$, where B is the box given by $0 \le x \le 1$, $1 \le y \le 2$ and $-1 \le z \le 1$.

(g) State and prove fundamental theorem for line integrals.

(h) Use Green's theorem to find the work done by the force field

$$\vec{F}(x, y) = (3y - 4x)\hat{i} + (4x - y)\hat{j}$$

when an object moves once counter-clockwise around the ellipse $4x^2 + y^2 = 4$.

(i) Let S be the hemisphere $x^2 + y^2 + z^2 = 4$ with $z \ge 0$. Evaluate the surface integral

$$\iint_S (x^2 + y^2) z dS$$

4. Answer any *two* of the following questions : 10×2=20

(a) (i) Find the equation for the tangent plane and the normal line at the point $P_0(1, -1, 2)$ on the surface S given by $x^2 y + y^2 z + z^2 x = 5$. 5

(ii) Evaluate $\int_0^\pi \int_0^2 \int_0^{\sqrt{4-r^2}} r \sin\theta \, dz \, dr \, d\theta$. 5

(b) (i) Given that the largest and the smallest values of

$$f(x, y) = 1 - x^2 - y^2$$

subject to the constraints $x + y = 1$ with $x \ge 0$, $y \ge 0$ exist, use Lagrange multipliers to find these extrema. 5

(ii) Define a conservative vector field. What is the scalar potential of a conservative vector field? Verify that the field $\vec{F} = 2xy\hat{i} + x^2\hat{j}$ is conservative, with scalar potential $f = x^2 y$. 2+1+2=5

(c) (i) Let $z = 4x - y^2$, where $x = uv^2$ and $y = u^3 v$. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$. 5

(ii) Compute $\iint_D \left(\frac{x-y}{x+y}\right)^4 dy\,dx$, where D is the rectangular region bounded by the line $x + y = 1$ and coordinate axes (using change of variables method). 5

5. Answer any *one* of the following questions : 14

(a) (i) Obtain the formula for converting a triple integral in rectangular coordinates to one in spherical coordinates. 7

(ii) A wire has the shape of the curve $x = \sqrt{2}\sin t$, $y = \cos t$, $z = \cos t$ for $0 \le t \le \pi$. If the wire has density

$$\delta(x, y, z) = xyz$$

at each point (x, y, z), then find its mass and centre of mass. 3+4=7

(b) (i) Use Stokes' theorem to evaluate the line integral $\oint \vec{F} \cdot \overrightarrow{dR}$, where

$$\vec{F} = (y^2 + z^2,\ x^2 + y^2,\ z^2 + x^2)$$

and C is the triangle (1, 0, 0), (0, 1, 0), (0, 0, 1) traversed in that order. 7

(ii) Find the area of the region D between $y = \cos x$ and $y = \sin x$ over the interval $0 \le x \le \frac{\pi}{4}$, using—

(1) single integral;

(2) double integral. 4+3=7

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