2022

(Held in 2023)

MATHEMATICS

Paper : MATHC3066

(Group Theory—I)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Choose the correct option from the following : $1\times6=6$

   (a) If a is a generator of a cyclic group G, then

      (i) $|G|\geq|a|$

      (ii) $|G|\leq|a|$

      (iii) $|G|=|a|$

      (iv) $|G|>|a|$

(b) The number of elements in the alternating group A_7 is

(i) 1530

(ii) 2520

(iii) 2410

(iv) 5040

(c) A homomorphism $f: G \to G'$ is one-one iff

(i) $\ker f = \{e\}$, where e is the identity element of G

(ii) $\ker f \neq \{e\}$, where e is the identity element of G

(iii) $\ker f = G$

(iv) $\ker f \neq G$

(d) If $(3, 7), (7, 9) \in U(8) \oplus U(10)$, then $(7, 9)\ (5, 7)$ is equal to

(i) $(3, 3)$

(ii) $(1, 3)$

(iii) $(1, 7)$

(iv) $(7, 7)$

(e) Let G be the group of non-zero real numbers under multiplication and $H = \{x \in G : x = 1 \text{ or } x \text{ is irrational}\}$. Then

(i) H is a normal subgroup of G

(ii) H is a cyclic subgroup of G

(iii) H is not a subgroup of G

(iv) None of the above

(f) If $\alpha = (1245)(36)$ is a permutation of degree 6, then the order of α is

(i) 1

(ii) 2

(iii) 3

(iv) 4

2. Answer the following questions : 2×5=10

(a) Define $*$ on $G = \{0, 1, 2\}$ by $a * b = |a - b|$. Examine if $(G, *)$ is a group.

(b) Prove that the inverse of an element in a group is unique.

(c) Check whether the element 2 is a generator of $\mathbb{Z}_8$.

(d) Find $U(9) \oplus U(7)$.

(e) If $\alpha = (12)(3)(45)$ and $\beta = (153)(24)$ are two elements of S_5, then find $\alpha\beta$.

3. Answer any *six* of the following : 5×6=30

(a) Let G be a group and $f: G \to G$ be defined as $f(x) = gxg^{-1}$, where g is a fixed element of G. Prove that f is a homomorphism. Also determine $\ker f$.

(b) State and prove Lagrange's theorem.

(c) In a finite cyclic group, prove that the order of an element divides the order of the group.

(d) Define centre of a group. Also show that the centre of a group is a subgroup of that group.

(e) Prove that a subgroup H of a group G is a normal subgroup of G if and only if $xHx^{-1} \subseteq H$ for all $x \in G$.

(f) Find the number of elements of order 5 in $\mathbb{Z}_{25} \oplus \mathbb{Z}_5$.

(g) Let G be a group and H is normal subgroup of G. Prove that the set $\frac{G}{H} = \{aH : a \in G\}$ is a group under the operation $(aH)(bH) = abH$, where $a, b \in G$.

(h) Let $f: G \to G'$ be an isomorphism and $a \in G$. Prove that $O(a) = O(f(a))$.

(i) Let H be a subgroup of a group G and $a, b \in G$. Show that $Ha = Hb$ if and only if $ab^{-1} \in H$.

4. Answer any *two* of the following : 10×2=20

(a) (i) State and prove the fundamental theorem of group homomorphism. 7

(ii) In $D_4 = \{R_0, R_{90}, R_{180}, R_{270}, H, V, D, D'\}$, find all elements $X \in D_4$ such that $X^3 = V$. 3

(b) (i) Let G be a cyclic group of order n. If d is a positive divisor n, then prove that the number of elements of order d in G is $\phi(d)$. 5

(ii) Make a list of all the subgroups of $\mathbb{Z}_{30}$. 5

(c) Prove Cauchy's theorems for finite Abelian groups.

5. Answer any *one* of the following : 14

(a) (i) If H and K are two subgroups of a group G, then show that HK is a subgroup of G, if and only if $HK = KH$. 4

(ii) If H and K are two subgroups of a group G, then prove that

$$O(HK) = \frac{O(H) \cdot O(K)}{O(H \cap K)}$$ 7

(iii) Determine the order of the permutation

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 1 & 7 & 8 & 6 \end{pmatrix}$$

Also check whether β is even or odd. 2+1=3

(b) *(i)* Prove that every group is isomorphic to a group of permutations. 8

(ii) If G and H are two finite cyclic groups, then prove that $G \oplus H$ is cyclic if and only if $O(G)$ and $O(H)$ are relatively prime. 6

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