5

63/1 (SEM-3) CC5/MATHC3056

2022

(Held in 2023)

MATHEMATICS

Paper : MATHC3056

(Theory of Real Functions)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Choose the correct answer from the following : 1×6=6

(a) Which of the following points is not a cluster point of the open interval]0, 1[?

(i) 0·1

(ii) 0·5

(iii) 1·0

(iv) 1·1

(b) What type of discontinuity the function $f(x) = \sin\left(\frac{1}{x}\right)$ has?

(i) Discontinuity of first kind

(ii) Discontinuity of second kind

(iii) Removable discontinuity

(iv) None of the above

(c) If $\lim_{x\to\infty} f(x) = l$ and $\lim_{x\to\infty} g(x)$ do not exist, then which of following about the limit $\lim_{x\to\infty} f(x)\cdot g(x)$ will be correct?

(i) $\lim_{x\to\infty} f(x)\cdot g(x)$ exist

(ii) $\lim_{x\to\infty} f(x)\cdot g(x) = l$

(iii) $\lim_{x\to\infty} f(x)\cdot g(x)$ does not exist

(iv) None of the above

(d) Which of the following functions is not a continuous function?

(i) A polynomial function

(ii) A rational function

(iii) A differentiable function

(iv) None of the above

(e) In which interval, Rolle's theorem is applicable for the function $f(x) = \sin x$?

(i) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $[0, \pi]$

(iii) $[-2\pi, 0]$

(iv) None of the above

(f) Which of the following statements is correct for the function $f(x) = x^3$?

(i) $f(x)$ has maximum value at $x = 0$

(ii) $f(x)$ has minimum value at $x = 0$

(iii) $f(x)$ has neither maximum value nor minimum value at $x = 0$

(iv) $f(x)$ has no point of inflexion

2. Answer the following questions : 2×5=10

(a) Find the cluster points of the following sets : 1×2=2

(i) $A = \{1, 2\}$

(ii) $B = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

(b) If a function f is continuous at a point c in an interval, then prove that the function $|f|$ is also continuous at the point c of the interval.

(c) Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ is a function. If $\exists$ a constant $K > 0$ such that $|f(x) - f(u)| < K|x - u|$ $\forall\, x, u \in A$, prove that the function f is uniformly continuous on A.

(d) Evaluate the value of θ that appears in Lagrange's mean value theorem $f(a+h) = f(a) + hf'(a+\theta h)$ for the function $f(x) = x^2 + 2x - 3$, given that $a = 1$ and $b = \frac{1}{2}$.

(e) State the conditions under which a function can be expanded as a Maclaurin's series.

3. Answer the following questions (any *six*) : 5×6=30

(a) If $f: A \to \mathbb{R}$ and x_0 is a cluster point of A, then prove that the following statements are equivalent :

(i) $\lim_{x \to x_0} f(x) = l$

(ii) Given any ε-nbhd. $V_\varepsilon(l)$ of l, there exists a δ-nbhd. $V_\delta(x_0)$ of x_0 such that if $x \neq x_0$ is any point in $V_\delta(x_0) \cap A$, then $f(x)$ belongs to $V_\varepsilon(l)$.

(b) Determine the constants a and b so that the function f defined below is continuous everywhere $f(x) = 2x+1$, if $x \le 1$, $f(x) = ax^2 + b$, if $1 < x < 3$ and $f(x) = 5x + 2a$, if $x \ge 3$.

(c) Examine the continuity and differentiability of the function $f: \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = |x-2| + |x-3|$$ 2+3=5

(d) Define uniform continuity of a function on an interval. Show that the function defined by $f(x) = x^3$ is uniformly continuous in $[-2, 2]$. 1+4=5

(e) If I be an interval and $f: I \to \mathbb{R}$ be continuous on I, then prove that the set $f(I)$ is an interval.

(f) Prove that between any two real roots of $e^x \sin x = 1$, there is at least one real root of $e^x \cos x + 1 = 0$.

(g) Show that

$$x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2}(1+x), \quad x > 0$$

(h) Prove, by using Lagrange's mean value theorem, that a function f is monotonically increasing at a point c if and only if $f'(c) \ge 0$.

(i) Find the maxima and minima of the function

$$f(x) = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x \quad \forall\, x \in [0, \pi]$$

4. Answer any *two* from the following questions : 10×2=20

(a) *(i)* State divergence criterion for limits. Use it to show that $\lim_{x \to 0} \sin(1/x)$ does not exist. 2+4=6

(ii) Show that $\sin x$ is uniformly continuous on $[0, \infty[$. 4

(b) State and prove Bolzano's intermediate value theorem. Use it to prove, if $I = [a, b]$ be a closed and bounded interval such that $f : I \to \mathbb{R}$ be continuous on I and $k \in \mathbb{R}$ be a number satisfying

$$\inf f(I) \le k \le \sup f(I)$$

then prove that $\exists$ a number $c \in I$ such that $f(c) = k$. 2+4+4=10

(c) Prove that the function f defined on an interval I is continuous at $a \in I$ if and only if for every sequence $\langle a_n \rangle$ in I

which converges to a

$$\lim_{n \to \infty} f(a_n) = f(a)$$

Use the above result to examine the continuity of the following function :

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is irrational} \\ -1, & \text{when } x \text{ is rational} \end{cases}$$

5+5=10

5. Answer any *one* from the following : 14

(a) Find the value of the following : 2+4+4+4=14

(i) $\lim_{x \to 0} \dfrac{x - \log(1+x)}{1 - \cos x}$

(ii) $\lim_{x \to 0} \dfrac{(1+x)^{1/x} - e}{x}$

(iii) $\lim_{x \to 0} \left(\dfrac{\tan x}{x}\right)^{1/x^2}$

(iv) a, b and c such that

$$\lim_{x \to 0} \frac{a + b\cos x + c\sin x}{x^2} = \frac{1}{2}$$

(b) *(i)* State and prove Taylor's theorem with Lagrange's form of remainder after n terms. 2+5=7

(ii) Prove, using Taylor's theorem, that $\log(1+x)$ lies between

$$x-\frac{x^2}{2} \text{ and } x-\frac{x^2}{2(1+x)} \quad \forall x$$ 3

(iii) If

$f(a+h)=$

$$f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a+\theta h)$$

$$0<\theta<1$$

and if f^{iv} is continuous and non-zero at $x=a$, show that

$$\lim_{h\to 0}\theta=\frac{1}{4}$$ 4

★★★