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63/1 (SEM-5) CC12/MATHC5126

2023

MATHEMATICS

Paper : MATHC 5126

(Group Theory–II)

Full Marks : 80

Pass Marks : 32

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Choose the correct answer : ***(any six)*** 1×6=6

   (i) Let $Q=\{\pm 1, \pm i, \pm j, \pm k\}$ be the Quaternion group with usual binary operation defined on Q. Then

   (A) $|Cl(k)|=2$

   (B) $|Cl(k)|=4$

   (C) $|Cl(k)|=1$

   (D) $|Cl(k)|=8$

Contd.

(ii) Let G and H be two finite groups such that $G \approx H$. Then

(A) $|G| = |H|$

(B) $G = H$

(C) $K \leq G \Rightarrow K \leq H$

(D) $|G| \neq |H|$

(iii) Let $G = \{e, a, b, c, \cdots\}$ be a group, and let $Inn(G) = \{\phi_e, \varphi_a, \varphi_b, \varphi_c, \cdots\}$. Then

(A) $\varphi_a = \varphi_b$ if and only if $a = b$

(B) $\varphi_a = \varphi_b$ even though $a \neq b$

(C) $\varphi_{ab} \neq \varphi_a \varphi_b$

(D) $(\varphi_{ab})^{-1} - \varphi_{a^{-1}b^{-1}}$

(iv) Let G and H be two groups. Then

(A) $Aut(G) \approx Aut(H)$ if and only if $G \approx H$

(B) $Aut(G) \approx Aut(H)$ even though $G \not\approx H$

(C) $|Aut(G)| = \infty$ if $|G| = \infty$

(D) $|Inn(G)| = |Aut(G)|$

(v) Which of the following is false?

(A) $U(90) \approx U(5) \oplus U(18)$

(B) $U(90) \approx U(2) \oplus U(9) \oplus U(5)$

(C) $U(90) \approx U(2) \oplus U(3) \oplus U(15)$

(D) $U(90) \approx U(9) \oplus U(10)$

(vi) Which of the following is not isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$?

(A) $\mathbb{Z}_2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_5$

(B) $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{15}$

(C) $\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$

(D) $\mathbb{Z}_2 \oplus \mathbb{Z}_{30}$

(vii) Let G be a finite abelian group such that $|G| = p^n m$, where p is a prime and $p \nmid m$. If $H = \{x \in G \mid x^{p^n} = e\}$ and $K = \{x \in G \mid x^m = e\}$, then which of the following is not true?

(A) $G = HK$

(B) $|H| = p^n$

(C) $H \cap K = \{e\}$

(D) $G \neq H \times K$

(viii) Let G be a finite group such that H is the only Sylow p-subgroup of G, where p is a prime. Then

(A) H is always cyclic

(B) Index of H is 2

(C) $|H|$ must be prime

(D) $|H|$ is equal to 1 modulo p

(ix) Let G be a group of order n. Let $a \in G$ be such that $|C(a)| = m$. Then which of the following is not correct?

(A) $|G : C(a)| = \frac{n}{m}$

(B) $|Cl(a)| = \frac{n}{m}$

(C) $|Cl(a)| \neq \frac{n}{m}$

(D) $|G : N(a)| = \frac{n}{m}$

(x) Let K_4 be the Klein's 4-group. Then the class equation of K_4 is

(A) $4 = 1 + 3$

(B) $4 = 1 + 1 + 1 + 1$

(C) $4 = 1 + 1 + 2$

(D) $4 = 2 + 2$

2. Answer the following questions : ***(any five)*** 2×5=10

(i) Let G be a group acting on a nonempty set X under the group action '.'. Then which of the following statement is true? Give justification.

(a) $g \cdot x_1 = g \cdot x_2 \Rightarrow x_1 = x_2 \;\; \forall\, g \in G$ and $\forall\, x_1, x_2 \in X$.

(b) $g_1 \cdot x_1 = g_2 \cdot x \Rightarrow g_1 = g_2 \;\; \forall\, g_1, g_2 \in G$ and $\forall\, x \in X$.

(ii) Show that the class equation of S_3 is $6 = 1 + 2 + 3$.

Contd.

(iii) Let G be an abelian group and φ a map defined on G such that $\varphi(x) = x^{-1}\ \forall\ x \in G$. Then show that φ is an automorphism.

(iv) Let G be a group such that $Cl(a) = \{a\}\ \ \forall\ a \in G$.
Then prove that G is abelian.

(v) Find the elements of order 2 in $\mathbb{Z}_8 \oplus \mathbb{Z}_2$.

(vi) Prove *or* disprove : $\mathbb{Z}_1 \oplus \mathbb{Z}_2 \approx \mathbb{Z}_2 \oplus \mathbb{Z}_1$.

(vii) Suppose that φ is an isomorphism from $\mathbb{Z}_3 \oplus \mathbb{Z}_5$ to $\mathbb{Z}_{15}$ and $\varphi(2, 3) = 2$. Find the element in $\mathbb{Z}_3 \oplus \mathbb{Z}_5$ that maps to 1 of $\mathbb{Z}_{15}$.

3. Answer the following questions : ***(any six)***
5×6=30

(i) Let $\mathbb{R}^2 = \{(a, b) \mid a, b \in \mathbb{R}\}$ be the group under componentwise addition. Define a map φ on $\mathbb{R}^2$ by $\varphi(a, b) = (b, a)$. Show that φ is an automorphism. How do you interpret φ geometrically?
4+1=5

(ii) Define a characteristic subgroup of a group. Prove that the centre of a group G is a characteristic subgroup of G.
1+4=5

(iii) Determine the number of cyclic subgroups of order 10 in $\mathbb{Z}_{100} \oplus \mathbb{Z}_{25}$.

(iv) If m divides the order of a finite abelian group G, then prove that G has a subgroup of order m.

(v) Let G be a group acting on a nonempty set X under the group action '$\cdot$', and let $Y \subseteq X$.
Let $G_Y = \{g \in G \mid g \cdot y = y\ \forall y \in Y\}$.
Show that G_Y is a subgroup of G.

(vi) Define a cycle type of a permutation. Write the cycle type of the permutations $\sigma_1 = (35)(2476)(89)$ and $\sigma_2 = (5629)(47)(81)$ on S_9. Are they conjugate? If so, find a τ on S_9 such that $\sigma_2 = \tau\sigma_1\tau^{-1}$.
1+1+1+2=5

Contd.

(vii) Let φ be an automorphism of a group G. Then show that $|\varphi(a)|=|a|$ for all $a \in G$. Hence, deduce that $|bab^{-1}|=|a|$ for all $a, b \in G$.

(viii) Define a commutator subgroup G' of a group G. If $G = S_3$, then find G'.
1+4=5

(ix) If G is a finite group and H a proper subgroup of G such that $|G|$ does not divide $|G:H|!$, then prove that H contains a nontrivial normal subgroup of G.

(x) Let G be a group such that $|G| = p^2$, where p is a prime. Then prove that G is abelian.

4. Answer the following questions : ***(any two)***
10×2=20

(a) (i) Let G and H be finite cyclic groups. Then prove that $G \oplus H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime. 5

(ii) Let $\mathbb{Z}$ be the additive group of integers. Define a map $* : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by $*(z, a) = z * a = z + a$ for all $z \in \mathbb{Z}$ and $a \in \mathbb{Z}$. Then prove that $*$ is a group action of $\mathbb{Z}$ on itself. When is a group action called faithful ? Is the action $*$ faithful ? Justify.
3+1+1=5

(b) (i) Define Kernel of a group action on a non-empty set. Show that Kernel of a group action is a subgroup of the group. 1+4=5

(ii) Let G be a nontrivial finite group whose order is a power of a prime p. Then prove that $|Z(G)| > 1$. 5

(c) (i) State and prove Generalized Cayley Theorem. 1+6=7

(ii) Let $\alpha = (12) \in S_3$. Then find $Cl(\alpha)$.
3

(d) Let $G = \{1, 8, 12, 14, 18, 21, 27, 31, 34, 38, 44, 47, 51, 53, 57, 64\}$ be a group under multiplication modulo 65. Show that $G \approx \mathbb{Z}_4 \oplus \mathbb{Z}_4$ and $G = \langle 8 \rangle \times \langle 12 \rangle$.

5. Answer the following questions : ***(any one)*** $14 \times 1 = 14$

(a) (i) Let G be a group, and let $Inn(G)$ be the linear automorphism group of G. Prove that

$$\frac{G}{Z(G)} \approx Inn\,(G)$$

5

(ii) State Sylow's Third Theorem. Apply this theorem to determine all the groups of order 66.

1+8=9

(b) (i) Define a simple group. Show that the group A_5 is simple. 1+6=7

(ii) Prove that for every positive integer n, $Aut\,(\mathbb{Z}_n) \approx U(n)$. 7

(c) (i) State and prove Cauchy theorem for a finite group. 1+9=10

(ii) Let $\mathbb{R}^+$ be the group of all positive real numbers under multiplication. Then show that the mapping φ defined on $\mathbb{R}^+$ by $\varphi(x) = \sqrt{x}\ \forall\ x \in \mathbb{R}^+$ is an automorphism of $\mathbb{R}^+$. 4