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63/1 (SEM–5) CC11/MATHC5116

2023

MATHEMATICS

Paper : MATHC 5116

(Multivariate Calculus)

Full Marks : 80

Pass Marks : 32

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following as directed : $1\times6=6$

(a) The Domain and Range of the function $f(x, y) = 3x + 5y + 2$ is

(i) Range of f is R and Domain of f is R^2

(ii) Range of f is R^2 and Domain of f is R

Contd.

(iii) Range of f is R and Domain of f is R

(iv) Range of f is $[3, 5]$ and Domain is $\{(x, y) \in R \mid 3x + 5y = 2\}$

(b) If $z = f(x, y) = 4x^2 + 3y^2$,
$x = x(t) = \sin t, \quad y = y(t) = \cos t$,
then $\frac{dz}{dt}$ is

(i) $4x\cos t - 3y\cos t$

(ii) $8x\sin t - 6y\cos t$

(iii) $8x\cos t - 6y\sin t$

(iv) $8y\sin t - 6x\cos t$

(c) If a double integral exists, then the two repeated integrals I_1 and I_2 is

(i) $I_1 > I_2$

(ii) $I_1 < I_2$

(iii) $I_1 = I_2$

(iv) None of the above

(d) The value of $\int_1^2 \int_0^3 x^2 y \, dx \, dy$ is

(i) $\frac{28}{2}$

(ii) $\frac{27}{2}$

(iii) $\frac{2}{27}$

(iv) $\frac{27}{3}$

(e) Let f be a real valued function of two variables. Let $D \subset R^2$ be the domain of the function and (a, b) an interior point of D such that f admits of second order continuous partial derivative in its neighbourhood. Suppose
$f_x(a, b) = 0 = f_y(a, b)$
and $A = f_{xx}(a, b), B = f_{xy}(a, b)$
and $c = f_{yy}(a, b)$.
Then the function f has local maximum value at (a,b), if

(i) $A > 0, \ AC - B^2 > 0$

(ii) $A < 0, \ AC - B^2 > 0$

(iii) $AC - B^2 < 0$

(iv) $AC - B^2 = 0$

(f) Existence of all the directional derivative at a point imply

(i) certainly continues

(ii) may not be continuous

(iii) does not exist

(iv) None of the above

2. Answer the following questions : ***(any five)*** 2×5=10

(a) Under what condition, in double integrals the order of integration can be changed at will ?

(b) Evaluate $\iint_R xy(x+y)\,dx\,dy$ where R is the region bounded by $x^2=y$ and $x=y$.

(c) If $f(x, y)=g(x^2-y^2, y^2-x^2)$ and g is differentiable, show that f satisfies the equation

$$y\frac{\partial f}{\partial x}+x\frac{\partial f}{\partial y}=0$$

(d) Find the equation of the tangent plane and normal line at the point $(-2, 1, -3)$ to the ellipsoid

$$\frac{x^2}{4}+y^2+\frac{z^2}{9}=3$$

(e) Evaluate by changing of integration

$$\int_0^\infty \int_0^x e^{-xy}\, y\,dy\,dx$$

(f) Find the extreme value of the function $f(x, y)=x^2+2y^2$ on circle $x^2+y^2=1$.

(g) If $z=x^2y+3xy^4$ where $x=\sin 2t$, and $y=\cos t$, find $\left.\frac{dz}{dt}\right|_{t=0}$

3. Answer the following questions : ***(any six)*** 5×6=30

(a) Show that the maximum and minimum of radii vector of the section of the surface

$$\left(x^2+y^2+z^2\right)^2=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$$

by the plane $\lambda x+\mu y+\gamma z=0$ are given by

$$\frac{a^2\lambda^2}{1-a^2\gamma^2}+\frac{b^2\mu^2}{1-b^2\gamma^2}+\frac{c^2\gamma^2}{1-c^2\gamma^2}=0$$

where $\gamma=$ radius vector.

(b) Examine the continuity at $(0,0)$ of function f :

$$f(x, y)=\begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} , & (x, y)\neq(0, 0) \\ 0 & (x, y)=(0, 0) \end{cases}$$

(c) If $z=f(x, y)$ is a differentiable function of x any y where $x=g(t)$ and $y=h(t)$ are both differentiable function of t. Then show that z is a differentiable function of t and

$$\frac{dz}{dt}=\frac{\partial t}{\partial x}\frac{dx}{dt}+\frac{\partial t}{\partial y}\frac{dy}{dt}$$

(d) Find the local maximum and minimum values and saddle point of

$$f(x, y)=x^4+y^4-4xy+1$$

(e) A rectangular box without a lid is to be made from $12cm^2$ of cardboard. Find the maximum volume of such a box.

(f) Integrate $\iint\limits_R (x-3y^2)\,dA$

where $R=\{(x, y)\mid 0\leq x\leq 2,\ 1\leq y\leq 2\}$

(g) Find the volume of the solid that the line under paraboloid $z=x^2+y^2$ and above the region D in the xy-plane bounded by the line $y=2x$ and parabola $y=x^2$.

(h) Show that a function which is differentiable at a point admits of partial derivative of that point.

(i) Evaluate $\iint x^{m-1}y^{n-1}\,dx\,dy$ over the region bounded by

$$x+y=h,\ x=0,\ y=0$$

(j) Evaluate

$$\int_{-2}^{2}\int_{\sqrt{4-x^2}}^{\sqrt{4+x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx$$

4. Answer ***any two*** of the following : 10×2=20

(a) (i) Suppose that the temperature at a point (x, y, z) in space is given by $T(x, y, z)=80/(1+x^2+2y^2+3z^2)$, where T is measured in degree Celsius and x, y, z in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase? 5

(ii) Evaluate

$$\int_0^a \int_{\frac{x}{a}}^{x} \frac{x}{x^2+y^2}\,dydx$$ 5

(b) (i) If $u = f(x, y)$ where $x = e^s \cos t$ and $y = e^s \sin t$.

Show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2s}\left[\frac{\partial^2 u}{ds^2} + \frac{\partial^2 u}{\partial t^2}\right]$$ 5

(ii) Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$. 5

(c) (i) Find the magnitude of the directional derivative along a line making 30° with positive direction of x-axis at for the function $\frac{x}{x^2+y^2}$. 5

(ii) Evaluate $\iiint_V \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ where V is the tetrahedron bounded by the plane $x = a$, $y = 0$, $x + y + z = 1$. 5

(d) (i) Let $z = 4x - y^2$ where $x = uv^2$ and $y = u^3 v$. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$. 5

(ii) Define a conservative vector field. What is the scalar potential of a conservative vector field? Verify that the field $\vec{F} = 2xy\hat{i} + x^2\hat{j}$ is conservative with scalar potential $f = x^2 y$. 5

5. Answer ***any one*** of the following questions : $14 \times 1 = 14$

(a) (i) State and prove Green's theorem in the plane. 10

(ii) Using Green's theorem evaluate $\oint_C y^2 dx + 3xy\,dy$ where C is the boundary of the semiangular region D in the half plane between the circle $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. 4

(b) (i) Evaluate $\iint_S F\,dS$ when

$$F(x, y, z) = xyi + \left(y^2 + e^{xz^2}\right)j + \sin(xy)\,k$$

and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the plane $z = 0$, $y = 0$ and $y + z = 2$. 7

(ii) Evaluate $\iint_S z\,dS$ where S is the surface whose side S_1 is given by the cylinder $x^2 + y^2 = 1$ whose bottom S_2 is the disk $x^2 + y^2 \le 1$ in the plane $z = 0$ and whose top S_3 is the part of the plane $z = 1 + x$ lies above S_2. 7

(c) (i) Obtain the formula for converting a triple integral in rectangular co-ordinates to one in spherical co-ordinates. 7

(ii) Find the area of the region D between $y = \cos x$ and $y = \sin x$ over the interval $0 \le x \le \frac{\pi}{4}$, using—

(1) single integral;

(2) double integral. 4+3=7
