

2018

MATHEMATICS

(Major)

Paper : 5.2

(Topology)

Full Marks : 60

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following questions : 1×7=7

(a) Describe the open spheres for any discrete metric space (X, d) .

(b) Find the derived sets of the following subsets of \mathbb{R} :

$$A =]0, 1], \quad B = \left\{ \frac{2n+1}{n} : n \in \mathbb{N} \right\}$$

$$C = \left\{ -\frac{1}{n} : n \in \mathbb{N} \right\}$$

(c) Define a Cauchy sequence in a metric space (X, d) .

(d) Define a topological space and give one example.

(e) Let

$$X = \{a, b, c\} \text{ and}$$

$$\mathcal{T} = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$$

is a topology on X . Find the derived set of $A = \{a, b\}$.

(f) Let \mathcal{T} be the topology on \mathbb{N} which consists of ϕ and all subsets of the form $G_m = \{m, m+1, m+2, \dots\}$, $m \in \mathbb{N}$. What are the open sets containing 4?

(g) What do you mean by a Banach space? Give one example.

2. Answer the following questions : 2×4=8

(a) Show that every closed interval is a closed set in the usual metric on \mathbb{R} .

(b) Let f be a mapping from \mathbb{R} into \mathbb{R} defined by

$$f(x) = \begin{cases} -2 & \text{when } x < 0 \\ 2 & \text{when } x \geq 0 \end{cases}$$

Examine whether f is continuous with respect to the usual topology on \mathbb{R} .

(c) Let $(X, \|\cdot\|)$ be a normed linear space and $x_n \rightarrow x$ and $y_n \rightarrow y$ in X . Show that $x_n + y_n \rightarrow x + y$.

(d) Prove the parallelogram law in an inner product space $(X, \langle \cdot, \cdot \rangle)$.

3. Answer the following questions : 5×3=15

(a) Let (X, d) be a metric space and A and B be subsets of X . Prove that—

(i) $A \subset B \Rightarrow D(A) \subset D(B)$

(ii) $D(A \cup B) = D(A) \cup D(B)$

(b) Let X be any set and \mathcal{F} be the collection of all those subsets of X whose complements are finite together with the empty set. Show that \mathcal{F} is a topology on X . What do you call this topology?

Or

Let (X, \mathcal{F}) be a topological space and $A \subset X$. Prove that $\bar{A} = A \cup D(A)$.

(c) Show that \mathbb{R}^n is a normed linear space with some suitable norm.

Or

Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space.

Prove that for all $x, y \in X$

$$4\langle x, y \rangle = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$$

4. Answer the following questions : 10×3=30

(a) Prove that every non-empty open set on the real line is the union of a countable class of pairwise disjoint open intervals.

Or

State and prove Cantor's intersection theorem for metric spaces.

- (b) Let (X, d) be a metric space and $x_0 \in X$ be fixed. Show that the real-valued function $f_{x_0}(x) = d(x, x_0)$, $x \in X$ is continuous. Is it uniformly continuous? Let (Y, P) be another metric space and $f: X \rightarrow Y$ be a mapping. Prove that f is continuous if and only if the inverse image of every open set in Y is an open set in X . 2+1+7

Or

Let X be a metric space and Y be a complete metric space. Let A be a dense subspace of X . If $f: A \rightarrow Y$ is uniformly continuous, then prove that f can be extended uniquely to a uniformly continuous mapping $g: X \rightarrow Y$.

- (c) Prove that a metric space is compact if and only if it is complete, and totally bounded.

Or

Let $\{A_\lambda: \lambda \in \Lambda\}$ be a family of connected subsets of a space X such that

$$\bigcap_{\lambda \in \Lambda} A_\lambda \neq \emptyset$$

Prove that $\bigcup_{\lambda \in \Lambda} A_\lambda$ is a connected set in X .
