2018

MATHEMATICS

(Major)

Paper : 4.1

(Real Analysis)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following as directed : 1×10=10

(a) Find the values of x and y, if

$$S=\{1-\frac{(-1)^n}{n}: n\in N\},\ \inf S=x \text{ and } \sup S=y$$

(b) Is the set Q of all rational numbers closed? Give justification.

(c) Define the limit inferior of the sequence $\{a_n\}$ of real numbers.

(d) If $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$, then the sequence $\{a_n b_n\}$ is always convergent.

(Write true or false)

(e) If the series

$$u_1 - u_2 + u_3 - u_4 + \cdots, (u_n > 0, \forall n)$$

is such that $u_{n+1} \leq u_n, \forall n$ and $\lim_{n \to \infty} u_n = 0$, then the series

(i) converges

(ii) diverges

(iii) oscillates

(Choose the correct answer)

(f) The series

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$$

is not convergent. Give reason.

(g) Evaluate :

$$\lim_{x \to 0+} \frac{\sin x}{\sqrt{x}}$$

(h) A function f is defined on R by

$$f(x) = \begin{cases} -x^2, & \text{if} \quad x \le 0 \\ 5x-4, & \text{if} \quad 0 < x \le 1 \\ 4x^2-3x, & \text{if} \quad 1 < x < 2 \\ 3x+4, & \text{if} \quad x \ge 2 \end{cases}$$

Discuss the kind of discontinuity at $x = 0$, if any.

(i) Find the value of $c \in]a, b[$ for Cauchy's mean value theorem for the functions $f(x) = e^x$ and $g(x) = e^{-x}$ in $[a, b]$.

(j) State the intermediate value theorem for derivatives.

2. Answer the following questions : $2 \times 5 = 10$

(a) If $a \in R$ such that $0 \le a < \varepsilon$ for every $\varepsilon > 0$, then prove that $a = 0$.

(b) Test the convergence of

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

(c) Show that the function $f(x) = \frac{1}{x}$ is not uniformly continuous on $]0, 1]$.

(d) Examine the function $(x-3)^5(x+1)^4$ for the extreme value $x=3$.

(e) Show that the function $f(x)=x^2$ is derivable on [0, 1].

3. Answer any *four* parts : 5×4=20

(a) Prove that the arbitrary intersection of closed sets is closed. 5

(b) If $\{a_n\}$ is any sequence of real numbers, then prove that

$$\inf a_n \le \underline{\lim} a_n \le \overline{\lim} a_n \le \sup a_n$$ 5

(c) If $\sum u_n$ is a positive term series, such that

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = A$$

then prove that the series converges if $A<1$. 5

(d) Test for convergence of the series

$$\sum \frac{1^2 \cdot 3^2 \cdots (2n-1)^2}{2^2 \cdot 4^2 \cdots (2n)^2} x^{n-1},\ x>0$$ 5

(e) Prove that if a function f is continuous on a closed interval $[a, b]$, then it attains its bounds at least once in $[a, b]$. 5

(f) Use Taylor's theorem to show that

$$\cos x \geq 1 - \frac{x^2}{2}$$

for all real $x \geq 0$. 5

4. Answer *either (a)* and *(b) or (c)* and *(d)* : 5×2=10

(a) If S_1, S_2 are subsets of R, then show that $(S_1 \cap S_2)' \subseteq S_1' \cap S_2'$. Give an example to show that the equality between $(S_1 \cap S_2)'$ and $S_1' \cap S_2'$ may not hold, where S_i' denote derived set of S_i for $i = 1, 2$. 3+2=5

(b) State and prove Sandwich theorem for sequence of real numbers. 5

(c) If $\{a_n\}$ is a sequence, such that $\lim \frac{a_{n+1}}{a_n} = l > 1$, then prove that $\lim a_n = \infty$. 5

(d) If the monotonic increasing sequence $\{S_n\}$ is bounded, then prove that it is convergent. 5

5. Answer *either* (*a*) and (*b*) *or* (*c*) and (*d*) : 5×2=10

(*a*) When is a series $\sum u_n$ said to be absolutely convergent? Show that for any fixed value of x, the series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$$

is convergent. 1+4=5

(*b*) State Gauss's test for convergence of a series. Applying this test, examine the convergence of the series

$$1+\frac{\alpha}{\beta}+\frac{\alpha(\alpha+1)}{\beta(\beta+1)}+\frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)}+\cdots\infty$$

where $\alpha > 0$ and $\beta > 0$. 1+4=5

(*c*) Using comparison test (first type), show that

$$1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$$

is convergent. 5

(*d*) Rearranging the terms of

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots\infty$$

show that the series can be made convergent to different limits. State a condition under which a series converges to the same limit after rearrangement. 4+1=5

6. Answer any *two* parts : 5×2=10

(a) When is a function $f(x)$ said to have a discontinuity of the first kind at $x = c$? If $[x]$ denotes the largest integer $\leq x$, then discuss the continuity at $x = 3$ of $f(x) = x - [x], \forall x \geq 0$.

Is the function continuous at the integral value $x = 2$? 1+3+1=5

(b) Prove the following : 2½+2½=5

(i) $\lim_{x \to c} f(x) = B$ implies $\lim_{x \to c} |f(x)| = |B|$

(ii) $\lim_{x \to 0} x \sin \frac{1}{x} = 0$

(c) Prove that a continuous and strictly increasing function f in $[a, b]$ is invertible and the inverse function is continuous in $[f(a), f(b)]$. 5

(d) Suppose that $f : R \to R$ is differentiable at c and that $f(c) = 0$. Show that $g(x) = |f(x)|$ is differentiable at c if and only if $f'(c) = 0$. 5

7. Answer any *two* parts : 5×2=10

(a) A twice differentiable function f is such that $f(a) = f(b) = 0$ and $f(c) > 0$, for $a < c < b$. Prove that there is at least one value λ between a and b for which $f''(\lambda) < 0$. 5

(b) Prove that between any two real roots of $e^x \sin x = 1$, there is at least one real root of $e^x \cos x + 1 = 0$. 5

(c) Show that $\sin x(1 + \cos x)$ is maximum at $x = \pi/3$. 5

(d) Find Maclaurin's power series expansion for the function

$$f(x) = \log(1 + x), \text{ for } -1 < x \le 1$$ 5

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