

2017

MATHEMATICS

( Major )

Paper : 5.2

( Topology )

Full Marks : 60

Time : 3 hours

*The figures in the margin indicate full marks  
for the questions*

1. Answer the following questions : 1×7=7

(a) Find the derived set of the sets  $A_1 = ]0, 1[$   
and  $A_2 = [0, 1]$  in the real line  $\mathbb{R}$  with the  
usual metric.

(b) Give an example to show that the  
intersection of an infinite family of open  
sets need not be open in a metric space.

(c) State Cantor's intersection theorem for  
metric spaces.

(d) Define the usual metric on  $\mathbb{R}$ .

(e) Give an example to show that  $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$ , in a topological space.

(f) Let

$$X = \{1, 2, 3, 4\} \text{ and}$$

$$\mathcal{T} = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}, \{2, 3, 4\}\}$$

Let  $f: X \rightarrow X$  be defined by  $f(1) = 2$ ,  $f(2) = 4$ ,  $f(3) = 2$ ,  $f(4) = 3$ . State whether  $f$  is continuous at 3 or not.

(g) Define a norm on the set  $\mathbb{C}^n$ .

2. Answer the following questions : 2×4=8

(a) In a metric space, every convergent sequence is a Cauchy sequence. Justify whether it is true or false.

(b) In the cofinite topological space  $(X, \mathcal{T})$ , find the closure of any subset  $A$  of  $X$ .

(c) Show that every inner product space is a normed linear space.

(d) Show that in a normed linear space  $(X, \|\cdot\|)$ ,  $|||x| - |y|| \leq \|x - y\| \forall x, y \in X$ .

3. Answer the following questions : 5×3=15

(a) Let  $(X, d)$  be a metric space. If  $x_0$  is a limit point of a subset  $A$  of  $X$ , then prove that there exists a sequence  $\{a_n\}$  of points of  $A$ , all distinct from  $x_0$ , which converges to  $x_0$ .

5

- (b) Let  $(X, \mathcal{T})$  be a topological space and  $Y$  be a nonempty subset of  $X$ . Prove that  $u = \{G \cap Y : G \in \mathcal{T}\}$  will be a topology on  $Y$ . Give the name of this topology. 4+1

Or

- Let  $X$  and  $Y$  be topological spaces and  $f$  be a bijective mapping of  $X$  to  $Y$ . Prove that  $f$  is continuous and open if and only if it is a homeomorphism. 5

- (c) If  $x$  and  $y$  are any two vectors in an inner product space  $(X, \langle \cdot, \cdot \rangle)$ , then prove that  $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$ . 5

Or

- In an inner product space  $(X, \langle \cdot, \cdot \rangle)$ , if  $x_n \rightarrow x$  and  $y_n \rightarrow y$ , then show that  $\langle x_n, y_n \rangle \rightarrow \langle x, y \rangle$ . 5

4. Answer the following questions : 10×3=30

- (a) Let  $C[a, b]$  denote the set of all real valued continuous functions defined on  $[a, b]$ . Prove that  $C[a, b]$  is complete with respect to a suitable metric defined on it.

Or

Let  $(X, d)$  be a metric space and  $A$  be a subset of  $X$ . Prove that—

(i)  $A$  is closed if and only if  $A$  contains all its limit points;

(ii) a point  $x \in X$  is a limit point of  $A$  if and only if every open sphere centred at  $x$  contains infinitely many points of  $A$ .

(b) State and prove Baire's category theorem for metric spaces.

Or

Prove that a metric space is second countable if and only if it is separable.

(c) Prove that a metric space is compact if and only if every collection of closed subsets of  $X$  with the finite intersection property has a nonempty intersection.

Or

Prove that a subspace of the real line  $\mathbb{R}$  is connected if and only if it is an interval.

\*\*\*