2017

MATHEMATICS

(Major)

Paper : 3.2

(Linear Algebra and Vector)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

GROUP—A

(Linear Algebra)

(*Marks* : 40)

1. Answer the following as directed : 1×7=7

(a) Let $U = \{(a, b, c) : a = b = c\}$ is a subset in $\mathbb{R}^3$. Determine whether or not U is a subspace of $\mathbb{R}^3$.

(b) If $P_n(t)$ be the vector space of all polynomials of degree $\leq n$, then dim $P_n(t)$ is

(i) $n-1$ *(ii)* n

(iii) $n+1$ *(iv)* n^2

(Choose the correct option)

(c) The set $\mathbb{R}$ of all real numbers is a vector space over the field $\mathbb{Q}$ of rational numbers. Examine whether or not the set $\{1, \sqrt{2}\}$ of vectors in $\mathbb{R}$ is linearly independent.

(d) Let $v = (1, 2, 3)$ and scalar $k = -3$, show that $T : R^3 \to R^2$ defined by $T(x, y) = (|x|, y + z)$ is not linear.

(e) If T is a linear operator, then the following are equivalent :

(i) A scalar λ is an eigenvalue of T

(ii) The linear operator $\lambda I - T$ is singular

(Write true or false)

(f) Let V be vector space over the field F and let T be a linear operator on V. Define the characteristic space associated with a characteristic value of T.

(g) Prove that for a linear operator (matrix) T, the scalar 0 is an eigenvalue of T if and only if T is singular.

2. Answer the following questions : 2×4=8

(a) Express $v = (2, -5, 3)$ in $\mathbb{R}^3$ as a linear combination of the vectors

$$u_1 = (1, -3, 2),\ u_2 = (2, -4, -1),\ u_3 = (1, -5, 7)$$

(b) Let $\mathbb{R}$ be the field of real numbers and V be the space of all functional from $\mathbb{R}$ into $\mathbb{R}$ which are continuous. Define T by

$$(Tf)(x) = \int_0^x f(t)\, dt$$

Show that T is a linear transformation from V into V.

(c) Consider the two bases of the vector space $\mathbb{R}^2(\mathbb{R})$:

$$B_1 = \{(1, 2), (3, 5)\} \text{ and } B_2 = \{(1, -1), (1, -2)\}$$

Find the change-of-basis matrix M from B_1 to the 'new' basis B_2.

(d) Using Cayley-Hamilton theorem, compute the inverse of

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$$

3. Answer any *one* part : 5

(a) Let V and W be vector spaces over the same field F and T be a linear transformation from V into W. Show that if V is finite dimensional, then

$$\text{rank }(T) + \text{nullity }(T) = \dim V$$

(b) Let V and W be vector spaces over the same field F and the linear mapping $T: V \rightarrow W$ is one-to-one and onto. Show that the inverse map $T^{-1}: W \rightarrow V$ is also linear.

4. Answer the following questions : 10×2=20

(a) When a vector space is said to be finitely generated? If V is a finitely generated vector space over a field F, prove that V has a finite basis and any two bases of V have same number of vectors.

Or

Suppose V is finite dimensional vector space over a field F and U is a subspace of V. Prove that there is a subspace W of V such that $V = U \oplus W$.

(b) State and prove Cayley-Hamilton theorem for the characteristic polynomial f of a linear operator T on a finite dimensional vector space V.

Or

Let P be the operator on $\mathbb{R}^2$ which projects each vector onto the x-axis, parallel to the y-axis, $P(x, y) = (x, 0)$. Show that P is linear. What is the minimal polynomial for P?

GROUP—B

(Vector)

(*Marks* : 40)

5. Answer the following : 1×3=3

(a) Find the constant p such that the vectors

$$\vec{a} = 2\vec{i} - \vec{j} + \vec{k},\ \vec{b} = \vec{i} + 2\vec{j} - 3\vec{k},\ \vec{c} = 3\vec{i} + p\vec{j} + 5\vec{k}$$

are coplanar.

(b) Prove that

$$(\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c}) \cdot \vec{d} = (\vec{a} \cdot \vec{d})[\vec{a}\ \vec{b}\ \vec{c}]$$

(c) If $\vec{a}$ and $\vec{b}$ lie in a plane normal to the plane containing $\vec{c}$ and $\vec{d}$, show that

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = 0$$

6. If $\vec{a}_1$, $\vec{b}_1$, $\vec{c}_1$ and $\vec{a}_2$, $\vec{b}_2$, $\vec{c}_2$ are reciprocal system of vectors, prove that

$$\vec{a}_1 \times \vec{a}_2 + \vec{b}_1 \times \vec{b}_2 + \vec{c}_1 \times \vec{c}_2 = \vec{0}$$ 2

7. Answer the following questions : 5×3=15

(a) If $\vec{a}$, $\vec{b}$, $\vec{c}$ are the position vectors of A, B, C, prove that

$$\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}$$

is a vector perpendicular to the plane ABC.

(b) *(i)* If $\frac{d\vec{p}}{dt} = \vec{u} \times \vec{p}$ and $\frac{d\vec{q}}{dt} = \vec{u} \times \vec{q}$

show that

$$\frac{d}{dt}(\vec{p} \times \vec{q}) = \vec{u} \times (\vec{p} \times \vec{q})$$

(ii) If $\vec{w} = 5t^2\vec{i} + t\vec{j} - t^3\vec{k}$, evaluate

$$\frac{d}{dt}(\vec{w} \cdot \vec{w})$$

3+2=5

(c) Prove that

$$\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2\vec{F}$$

8. Answer the following questions : 10×2=20

(a) If $\vec{c}$ is a constant vector, prove that

$$\operatorname{div}(r^n(\vec{c}\times\vec{r}))=0,$$

where $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$.

Or

Prove that the necessary and sufficient condition for a vector $\vec{v}(t)$ to have a constant direction is

$$\vec{v}\times\frac{d\vec{v}}{dt}=\vec{0}$$

(b) Evaluate

$$\int_C \vec{F}\cdot d\vec{r}$$

where $\vec{F}(x, y, z)=yz\vec{i}-xz\vec{k}$ and C is the line segment from (−1, 2, 0) and (3, 0, 1).

Or

Find the work done when a force

$$\vec{F}=(x^2-y^2+x)\vec{i}-(2xy+y)\vec{j}$$

moves a particle in xy-plane from (0, 0) to (1, 1) along the parabola $y^2=x$.

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