2016

MATHEMATICS

(Major)

Paper : 5.2

(Topology)

Full Marks : 60

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following questions : 1×7=7

(a) Give an example to show that in a metric space, a Cauchy sequence may not always be convergent.

(b) Find the derived set, interior and closure of the set

$$A = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$$

in the real line $\mathbb{R}$ with the usual metric.

(c) For $x, y \in \mathbb{R}$, define $d(x, y) = |x^2 - y^2|$. Examine whether d is a metric on $\mathbb{R}$.

(d) Let $X = \{a, b, c\}$. Which of the following sets is not a topology on X?

(i) $\{\phi, X\}$

(ii) $\{\phi, \{a\}, X\}$

(iii) $\{\phi, \{a\}, \{b\}, \{a, b\}, X\}$

(iv) $\{\phi, \{a\}, \{b\}, \{c\}, X\}$

(e) Let X be any set and let $\mathscr{T} = \{\phi, A, B, X\}$, where A and B are non-empty disjoint proper subsets of X. Find the conditions A and B must satisfy in order that $\mathscr{T}$ will be a topology on X.

(f) Show that the usual metric on $\mathbb{Z}$ (the set of all integers) induces the discrete topology for $\mathbb{Z}$.

(g) Define Hilbert space and give an example.

2. Answer the following questions : $2 \times 4 = 8$

(a) Every subset of a discrete metric space is closed. Justify whether it is true or false.

(b) Let $X = \{a, b, c, d, e\}$ and let $S = \{\{a, b\}, \{b, c\}, \{a, d, e\}\}$. Find the topology on X generated by S.

(c) Show that every normed linear space is a metric space.

(d) Prove the parallelogram law in an inner product space $(X, \langle \cdot, \cdot \rangle)$.

3. Answer the following questions : 5×3=15

(a) Let (X, d) be a complete metric space and let Y be a subspace of X. Prove that Y is complete if and only if Y is closed.

(b) Let $(X, \mathscr{T})$ be a topological space and $A \subset X$. Show that A is closed if and only if A contains each of its limit points.

Or

Prove that a mapping f from a topological space X into another topological space Y is continuous if and only if $f(\bar{A}) \subset \overline{f(A)}$ for every set $A \subset X$.

(c) Show that $\mathbb{C}^n$ is a Banach space.

Or

Let $(X, <\cdot, \cdot>)$ be an inner product space. Prove that for all $x, y \in X$,

$$4\langle x, y\rangle = ||x+y||^2 - ||x-y||^2 + i\,||x+iy||^2 - i\,||x-iy||^2$$

4. Answer the following questions : 10×3=30

(a) Prove that the set $\mathbb{R}^n$ of n-tuples $x = (x_1, x_2, \cdots, x_n)$ of real numbers is a complete metric space with respect to the usual metric. 10

Or

State and prove Cantor's intersection theorem for metric spaces. 10

(b) Prove that in a metric space (X, d), the union of a finite number of nowhere dense sets is nowhere dense. Again, if A is nowhere dense in X, then show that each open sphere in X contains a closed sphere which contains no point of A.

6+4=10

Or

Let X be a metric space and Y be a complete metric space. Let A be a dense subspace of X. If $f : A \to Y$ is uniformly continuous, then prove that f can be extended uniquely to a uniformly continuous mapping $g : X \to Y$. 10

(c) Prove that a subset A of $\mathbb{R}$ is compact if and only if it is closed and bounded. 10

Or

Prove that the continuous image of a connected metric space is connected. Also show that the range of a continuous real-valued function defined on a connected space is an interval. 10

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