2016

MATHEMATICS

(Major)

Paper : 3.1

(Abstract Algebra)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following as directed : 1×10=10

(a) If H is a subgroup of S_n $(n \geq 2)$ contains both odd and even permutations, then $O(H)$ is even.

(Justify whether it is True or False)

(b) Which of the following is true?

If $G/H \cong G/K$, then $H = K$ when

(i) G is not a cyclic group

(ii) G is a cyclic group

(iii) G is a permutation group

(c) State first fundamental theorem on isomorphism of group.

(d) In a Boolean ring R, every prime ideal $P \neq R$ is maximal ideal.

(Justify whether it is True or False)

(e) Define simple ring.

(f) Let R be a commutative ring with unity and let M be a maximal ideal of R such that $M^2 = \{0\}$. If N is any maximal ideal of R, then $N \neq M$.

(Justify whether it is True or False)

(g) Define centre of a group G.

(h) Any finite p-group has non-trivial centre.

(Justify whether it is True or False)

(i) Which of the following is true?

(i) A PID is a Euclidean domain.

(ii) A Euclidean domain is a PID.

(iii) A field is not a PID.

(j) Give examples of two zero divisors in the ring $M_{2\times 2}$, the set of all 2×2 matrices.

2. Answer the following questions : 2×5=10

(a) Verify with an example that union of two subspaces of a vector space may not be a subspace.

(b) Let $f: G \to G'$ be a group homomorphism. Show that f is one-one if $\ker f = \{e\}$, where $\ker f$ is the kernel of f and $e \in G$ being identity element.

(c) Let G be a finite group. Show that G is a p-group (p-prime) if $O(G) = p^n$.

(d) If D is an integral domain and if $na = 0$ for some $0 \neq a \in D$ and some integer $n \neq 0$, then show that the characteristic of D is finite.

(e) Let R be a commutative ring and P a prime ideal of R. Then show that R / P is an integral domain.

3. Answer the following questions : 5×4=20

(a) Let $G = (a)$ and $G' = (b)$ be two cyclic groups of same order. Define $\phi : G \to G'$ by $\phi(a^r) = b^r$ for all integers r. Show that ϕ is an isomorphism.

Or

Let H and K be two normal subgroups of a group G such that $H \subseteq K$. Then prove that

$$\frac{G}{K} \cong \frac{G / H}{K / H}$$

(b) Prove that every finite integral domain is a field. Is it true for infinite integral domain? Justify your answer.

Or

Define idempotent and nilpotent elements of a ring. Show that a non-zero idempotent element cannot be nilpotent.

(c) If G is a finite group such that $O(G) = p^2$, where p is prime, then show that G is Abelian.

(d) If F is a field, then prove that the polynomial ring $F[x]$ over F is a Euclidean ring.

4. Answer the following questions : $10 \times 4 = 40$

(a) State and prove Cayley's theorem. $2+8=10$

Or

Let H be a normal subgroup of a group G. Then show that there exists a one-one onto mapping from X, the set of all subgroups of G containing H and Y, the set of all subgroups of G/H. 10

(b) In a ring R, the equation $ax = b (a \neq 0)$ has a solution, then show that R is a division ring. Also prove that the centre $Z(R)$ of a division ring R is a field. 6+4=10

Or

Prove that an ideal M of a commutative ring R with unity is a maximal ideal if and only if R / M is a field. Examine whether $Z / \langle 4 \rangle$ is a field or not. 7+3=10

(c) If $Z(G)$, $\text{Inn}(G)$ and $\text{Aut}(G)$ are respectively centre of G, inner automorphism of G and automorphism of G, then show that

$$\text{Inn}(G) \trianglelefteq \text{Aut}(G) \text{ and } G / Z(G) \cong \text{Inn}(G)$$

5+5=10

Or

If G is a finite group and $p \mid O(G)$, where p is prime, prove that a Sylow p-subgroup H of G is normal subgroup of G if and only if H is the only Sylow p-subgroup of G. Further, using Sylow's theorem, prove that no group of order 30 is simple. 5+5=10

(d) Define Euclidean domain. Show that the ring of integers $\mathbb{Z}$ is a Euclidean domain. Prove that every ideal in a Euclidean domain is a principal ideal. 2+4+4=10

Or

Find the field of quotients of the integral domain

$$\mathbb{Z}[i] = \{a + ib : a, b \in \mathbb{Z}\}$$ 10

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