2015

MATHEMATICS

(Major)

Paper : 5.2

(Topology)

Full Marks : 60

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following questions : $1\times7=7$

(a) Describe the open sphere of unit radius about (0, 0) for the following metric on $\mathbb{R}^2$:

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2|,$$
$$x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{R}^2$$

(b) Consider $\mathbb{R}$ with the usual metric. Find the derived set of each of the following subsets of $\mathbb{R}$:

$$A =]0, 1[$$
$$B = \left\{1 + \frac{1}{n} : n \in \mathbb{N}\right\}$$
$$C = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$$

(c) A finite set in any metric space has no limit point.
Justify whether it is true or false.

(d) Give an example to show that the union of two topologies on a set may not be again a topology.

(e) Let $\mathscr{T}$ be the topology on $\mathbb{N}$ which consists of ϕ and all subsets of the form $G_m = \{m, m+1, m+2, \ldots\}$, $m \in \mathbb{N}$. What are the open sets containing 5?

(f) Consider the topology $\mathscr{T} = \{\phi, \{a\}, \{b, c\}, X\}$ on $X = \{a, b, c\}$ and the topology $\mathscr{U} = \{\phi, Y, \{r\}, \{p, q\}\}$ on $Y = \{p, q, r\}$. Let $f : (X, \mathscr{T}) \to (Y, \mathscr{U})$ be a mapping defined by $f(a) = p$, $f(b) = q$ and $f(c) = r$. State whether f is a homeomorphism or not.

(g) Define an inner product space and give an example.

2. Answer the following questions : $2 \times 4 = 8$

(a) Give an example to show that in a metric space union of an infinite number of closed sets may not be a closed set.

(b) Let $X = \{1, 2, 3, 4\}$ and $A = \{\{1, 2\}, \{2, 4\}, \{3\}\}$. Determine the topology on X generated by A as a subbase and hence determine the base for this topology.

(c) Every inner product space is a normed linear space.
Justify whether it is true or false.

(d) Let $(X, \|\cdot\|)$ be a normed linear space and $x_n \to x$ and $y_n \to y$ in X. Show that

$$x_n + y_n \to x + y$$

3. Answer the following questions : 5×3=15

(a) Let (X, d) be a metric space and A be a subset of X. If x is a limit point of A, prove that there exists a sequence $\langle a_n \rangle$ of points of A, all distinct from x, which converges to x.

(b) Let $(X, \mathscr{T})$ be a topological space and $A \subset X$. Prove that the union of A and the set of its accumulation points is closed.

Or

Let $(X, \mathscr{T})$ be a topological space and $\langle f_n \rangle$ be a sequence of complex valued functions defined on X which converges uniformly to a function f defined on X. Prove that if all f_n's are continuous, then f is also continuous.

(c) Show that $\mathbb{R}^n$ is a normed linear space with some suitable norm.

Or

If x and y are any two vectors in an inner product space $(X, \langle \cdot, \cdot \rangle)$, then prove that $|\langle x, y \rangle| \le \| x \| \cdot \| y \|$.

4. Answer the following questions : 10×3=30

(a) Let $C[a, b]$ be the set of all real valued continuous functions defined on $[a, b]$. For $f, g \in C[a, b]$, define

$$d(f, g) = \sup_{x \in [a, b]} |f(x) - g(x)|$$

Show that with respect to this metric, $C[a, b]$ is a complete metric space. 10

Or

Let (X, d) be a metric space. Show that the mapping $d_1 : X \times X \to \mathbb{R}$ defined by

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is also a metric on X and that d and d_1 are equivalent. 10

(b) State and prove Baire's Category theorem for metric spaces. 10

Or

Let (X, d) be a metric space and $x_0 \in X$ be fixed. Show that the real valued function $f_{x_0}(x) = d(x, x_0)$, $x \in X$ is continuous. Is it uniformly continuous? Let (Y, ρ) be another metric space and $f : X \to Y$ be a mapping. Prove that f is continuous if and only if the inverse image of every open set in Y is an open set in X. 2+1+7=10

(c) Prove that a metric space is sequentially compact if and only if it has the Bolzano-Weierstrass property. 10

Or

Define disconnected metric space and give an example. Let (X, d) be a metric space and A be a connected subset of X such that $A \subseteq B \subseteq \overline{A}$. Prove that B is connected and hence deduce that $\overline{A}$ is connected. 2+8=10

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