

2015

MATHEMATICS

( Major )

Paper : 5.1

( Real and Complex Analysis )

Full Marks : 60

Time : 3 hours

*The figures in the margin indicate full marks  
for the questions*

1. Answer the following questions : 1×7=7
- (a) State a sufficient condition for differentiability of a function  $f$  at a point  $(a, b)$  of the domain of definition.
  - (b) What is meant by extreme values of a function of  $n$  variables?
  - (c) State one limitation of the Riemann integral.
  - (d) Give a simple application of Mean-value theorem of integral calculus.
  - (e) Define a convergent improper integral.
  - (f) What do you call that particular mapping which projects a sphere onto a plane?

(g) Give an example of critical point of a differentiable function of a complex variable.

2. Answer the following questions : 2×4=8

(a) Show that the function  $f$  defined by

$$f(x) = \begin{cases} 0, & \text{when } x \text{ is rational} \\ 1, & \text{when } x \text{ is irrational} \end{cases}$$

is not Riemann integrable on any interval.

(b) Test the convergence of the following improper integral

$$\int_0^1 \frac{dx}{(1-x)^{1/2} (1+x+x^2)^{1/2}}$$

(c) Show that the function  $f(z) = xy + iy$  is everywhere continuous but is not analytic.

(d) Evaluate

$$\frac{1}{2\pi i} \oint_C \frac{e^z}{z-2}$$

where  $C$  is the circle  $|z|=1$ .

3. Answer any *three* parts : 5×3=15

(a) Prove that the function

$$f(x, y) = \sqrt{|xy|}$$

is not differentiable at the point  $(0, 0)$ , but that both  $f_x$  and  $f_y$  exist thereat.

(b) State and prove the first Mean-value theorem of integral calculus.

(c) Let  $f$  and  $g$  be two positive functions such that  $f(x) \leq g(x)$ , for all  $x \in [a, b]$ . Then compare the two integrals  $\int_a^b f dx$  and  $\int_a^b g dx$  for convergence and divergence with justification.

(d) If  $f(z)$  is an analytic function of  $z$ , prove that

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |\operatorname{Re} f(z)|^2 = 2 |f'(z)|^2$$

(e) If  $f(z)$  is analytic inside and on the boundary  $C$  of a simply-connected region  $R$ , prove that

$$\oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

4. Answer either (a) or (b) :

(a) (i) Let  $f_y$  exists in a certain neighbourhood of a point  $(a, b)$  of the domain of definition of a function  $f$ , and  $f_{yx}$  is continuous at  $(a, b)$ . Then prove that  $f_{xy}(a, b)$  exists, and is equal to  $f_{yx}(a, b)$ .

- (ii) If  $u = \frac{x^2 + y^2 + z^2}{x}$ ,  $v = \frac{x^2 + y^2 + z^2}{y}$   
and  $w = \frac{x^2 + y^2 + z^2}{z}$ , then show  
that

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{x^2 y^2 z^2}{(x^2 + y^2 + z^2)^3} \quad 5$$

- (b) (i) If  $\phi$  is bounded and monotonic in  $[a, \infty[$ , and  $\int_a^\infty f dx$  is convergent at  $\infty$ , then prove that  $\int_a^\infty f \phi dx$  is convergent at  $\infty$ . 6

- (ii) State Dirichlet's test for convergence of improper integrals. Use it to test the convergence of the following integral for  $p > 0$ ,

$$\int_1^\infty \frac{\sin x}{x^p} dx \quad 1+3=4$$

5. Answer either (a) or (b) :

- (a) Prove or disprove the following two statements :

- (i) A bounded function  $f$ , having a finite number of points of discontinuity on  $[a, b]$ , is Riemann integrable on  $[a, b]$ . 5

(ii) A bounded function  $f$ , having an infinite number of points of discontinuity on  $[a, b]$ , is not Riemann integrable on  $[a, b]$ . 5

(b) (i) Show that the function  $[x]$ , where  $[x]$  denotes the greatest integer not greater than  $x$ , is Riemann integrable on  $[0, 3]$ , and  $\int_0^3 [x] dx = 3$ . 5

(ii) Show that  $f(x, y) = y^2 + x^2y + x^4$  has a minimum at  $(0, 0)$ . 5

6. Answer either (a) or (b) :

(a) (i) Examine the nature of the function

$$f(z) = \frac{x^2y^5(x+iy)}{x^4+y^{10}}, \quad z \neq 0, \quad f(0) = 0$$

in a region including the origin. 5

(ii) Show that a bilinear transformation can be considered as combination of the transformations of translation, rotation, stretching and inversion. 5

(b) (i) Show that the function  $u = x^3 - 3xy^2$  is harmonic and find the corresponding analytic function. 5

(ii) Show that the inverse of a point  $a$  with respect to the circle  $|z - c| = r$  is the point  $c + \frac{r^2}{\bar{a} - \bar{c}}$ . 5

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