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3 (Sem 4) MAT M1

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2015

MATHEMATICS

(Major)

Theory Paper : M-4.1

(Real Analysis)

Full Marks – 80

Time – Three hours

The figures in the margin indicate full marks
for the questions.

1. Answer the following as directed : $1 \times 10 = 10$
- (a) Give an example of a set which is not an interval but is a neighbourhood of each of its points.
- (b) Define an open subset of real numbers.

[Turn over

(c) The set of limit points of $\{1, 3, 5, 7, 9\}$ is

(i) $\{1, 3\}$

(ii) $\{7, 9\}$

(iii) $\{1, 3, 5, 9\}$

(iv) None of these

(Choose the correct answer)

(d) Write whether the following statement is true or false :

A sequence having only one limit point is convergent.

(e) The sequence $\left\{ \frac{(-1)^n}{n} \right\}$ is

(i) Convergent

(ii) Divergent

(iii) Oscillates finitely

(iv) Oscillates infinitely

(Choose the correct answer)

(f) Fill in the blank :

If a function f is derivable on a closed interval $[a, b]$ and $f'(a) < 0$ and $f'(b) > 0$ then there exists at least one point c between a and b such that $f'(c) = \underline{\hspace{2cm}}$.

(g) If $f(x) = \begin{cases} x & , 0 < x < 1 \\ 3 - x & , 1 \leq x \leq 2, \end{cases}$ then

(I) $\lim_{x \rightarrow 1^-} f(x) = 1$

(II) $\lim_{x \rightarrow 1^+} f(x) = 2$

(III) $\lim_{x \rightarrow 1} f(x) = 2$

(IV) $\lim_{x \rightarrow 1} f(x) = 1$

Of these statements

- (i) I and III are correct
- (ii) II and IV are correct
- (iii) I and II are correct
- (iv) III alone is correct

(Choose the correct answer)

(h) Find the maximum value of $\sin x + \cos x$.

(i) Evaluate : $\lim_{x \rightarrow 0} \frac{1 - \cos x}{3x^2}$

(j) The value of 'C' in Lagrange's Mean Value theorem for $f(x) = \alpha x^2 + \beta x + \gamma$, $\alpha \neq 0$ in $[a, b]$ is given by

(i) $\frac{a+b}{2}$ (ii) \sqrt{ab}

(iii) $\frac{2ab}{a+b}$ (iv) $\frac{a}{b} + \frac{b}{a}$

(Choose the correct answer)

2. Answer the following questions : $2 \times 5 = 10$

(a) Show that the following set

$\left\{1, -1, 1\frac{1}{2}, -1\frac{1}{2}, 1\frac{1}{3}, -1\frac{1}{3}, \dots\right\}$ is closed
but not open.

(b) Show that the series $\sum \frac{(-1)^{n+1}}{n^p}$ is absolutely
convergent for $p > 1$, but conditionally
convergent for $0 < p \leq 1$.

(c) Examine the continuity at $x=0$ of the function $f(x) = [x] - [-x]$ where $[x]$ denotes the largest integer $\leq x$.

(d) Verify Cauchy's Mean Value theorem for the functions $f(x) = \sin x$ and $g(x) = \cos x$ in $\left[-\frac{\pi}{2}, 0\right]$.

(e) Use Taylor's theorem to show that $\cos x \geq 1 - \frac{x^2}{2}$ for all $x \geq 0$.

3. Answer any *four* parts : 5×4=20

(a) Prove that a set is closed if and only if its complement is open. 5

(b) Show that

(i) $\lim_{n \rightarrow \infty} \left[\frac{1}{n^2} + \frac{1}{(n+1)^2} + \dots + \frac{1}{(2n)^2} \right] = 0$

(ii) $\lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1$ 5

- (c) State the Leibnitz test for convergence of an alternating series. Applying the test show that the series

$$1 - \frac{1}{3 \cdot 2^2} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 4^2} + \dots \text{ is convergent.}$$

1+4=5

- (d) Show that the series is convergent 5

$$\frac{1 \cdot 2}{3^2 \cdot 4^2} + \frac{3 \cdot 4}{5^2 \cdot 6^2} + \frac{5 \cdot 6}{7^2 \cdot 8^2} + \dots$$

(Use Comparison test)

$$(e) \text{ If } f(x) = \begin{cases} x \left(e^x - e^{-x} \right) & , x \neq 0 \\ \frac{1}{e^x + e^{-x}} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

show that f is continuous but not derivable at $x=0$. 5

- (f) (i) Find the values of a and b in order that

$$\lim_{x \rightarrow 0} \frac{x(1 + a \cos x) - b \sin x}{x^3} = 1$$

(ii) Evaluate : $\lim_{x \rightarrow 0} \left(\frac{1}{e^x - 1} - \frac{1}{x} \right)$ 3+2=5

4. Answer either (a) and (b) or (c) and (d) : $5 \times 2 = 10$

(a) Prove that every infinite bounded set has a limit point.

Can an infinite unbounded set have a limit point? Justify your answer. $4+1=5$

(b) State Cauchy's General Principle of convergence of a sequence. Using this show that the following sequence $\{S_n\}$ where

$$S_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} \text{ is convergent.}$$

$$1+4=5$$

(c) Prove that a convergent sequence of real numbers is bounded. Is the converse true? Justify your answer. $3+2=5$

(d) Show that the sequence $\{S_n\}$ defined by recursion formula $S_1 = \sqrt{2}, S_{n+1} = \sqrt{2S_n}$ converges to 2. 5

5. Answer either (a) and (b) or (c) and (d) : $5 \times 2 = 10$

(a) Prove that a necessary condition for convergence of an infinite series $\sum_{n=1}^{\infty} u_n$ is

$$\lim_{n \rightarrow \infty} u_n = 0$$

With an example show that it is not a sufficient condition. $3+2=5$

- (b) State Raabe's test for convergence of a series. Applying this test, examine the convergence of the series 1+4=5

$$1 + \frac{3}{7}x + \frac{3}{7} \cdot \frac{6}{10}x^2 + \frac{3}{7} \cdot \frac{6}{10} \cdot \frac{9}{13}x^3 + \dots$$

- (c) Applying Logarithmic test prove that the series $1 + \frac{1!}{2}x + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + \dots$ converges if $x < e$ and diverges if $x \geq e$. 5

- (d) Test the convergence of the series

$$1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot 2\cdot 3\cdot\gamma(\gamma+1)(\gamma+2)}x^3 + \dots$$

for all positive values of x ; α, β, γ being all positive. 5

6. Answer the following questions :

- (a) Prove that if a function is continuous in a closed interval, then it is bounded therein. 5

Or

Show that the function f defined by

$$f(x) = \begin{cases} -x, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}$$

is continuous only at $x = 0$.

5

- (b) Define uniform continuity of a function on an interval.

Prove that every uniformly continuous function on an interval is continuous on that interval.

Justify with an example that the converse is not true.

$$1+2+2=5$$

Or

- (c) Show that the function $f(x) = \frac{1}{x^2}$ is uniformly continuous on (a, ∞) , where $a > 0$ but not uniformly continuous on $(0, \infty)$. 5

7. Answer any two parts :

- (a) State Rolle's theorem. Using it prove that if $f'(x)$ and $g'(x)$ exist for all $x \in [a, b]$, and $g'(x) \neq 0$ for all $x \in (a, b)$, then for some

$$c \text{ between } a \text{ and } b, \frac{f(c) - f(a)}{g(b) - g(c)} = \frac{f'(c)}{g'(c)}$$

$$1+4=5$$

(b) Show that

$$\frac{v-u}{1+v^2} < \tan^{-1} v - \tan^{-1} u < \frac{v-u}{1+u^2}, \text{ if } 0 < u < v.$$

$$\text{Hence deduce that } \frac{\pi}{4} + \frac{3}{25} < \tan^{-1} \frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}.$$

$$3+2=5$$

(c) Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius a is $\frac{2a}{\sqrt{3}}$. 5

(d) Find Maclaurin's power series expansion for the function

$$f(x) = \log(1+x) \text{ for } -1 < x \leq 1. \quad 5$$