Total No. of printed pages = 7

3 (Sem 3) MAT M1

2015

MATHEMATICS

(Major)

Paper : 3.1

(Abstract Algebra)

Full Marks – 80

Time – Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following as directed : 1×10=10

(a) Let G be a group and $f : G \to G$ such that $f(x) = x^{-1}$ be a homomorphism. Then G is abelian.

–Justify whether it is true or false.

(b) Let G and G′ be two groups and $\phi : G \to G'$ be a homomorphism. If $a \in G$ and $o(a)$ is finite then examine whether $o(\phi(a))$ is a divisor of $o(a)$.

[Turn over

(c) Let G be a group and N be a normal subgroup of G. Define the canonical homomorphism from G to the quotient group G/N.

(d) Choose the correct option :

If the characteristic of an integral domain D is a non-zero number p, then the order of every non-zero element in the group (D, +) is

(i) p + 1 (ii) p

(iii) p − 1 (iv) none of these

(e) If G is an infinite cyclic group, then Aut (G) (the set of all automorphisms on G) is a group of order 2.

–State whether true or false.

(f) Define inner automorphism of a group G.

(g) If K is the only Sylow p-subgroup of a group G, then K is normal in G.

–Justify whether it is true or false.

(h) Define a ring homomorphism on the ring of complex numbers.

(i) Give an example of the principal ideal domain.

(j) Let I be an ideal of a ring R. If R is commutative, then R/I may not be commutative.

—State whether it is true or false.

2. Answer the following questions : $2\times5=10$

(a) Let G be a group of order 10 and G' be a group of order 6. Prove that there does not exist a homomorphism of G onto G'.

(b) If a is an invertible element in a ring R with unity, then show that a is not a divisor of zero.

(c) If U and W are two subspaces of a vector space V, then show that $U+W$ is the smallest subspace of V containing the subspaces U and W.

(d) Show that a group of order p^3, where p is a prime, may not be an abelian group.

(e) Let R be a ring with unity 1 and f is a homomorphism of R into an integral domain R′. If Ker f ≠ R, prove that f(1) is the unity of R′.

3. Answer the following questions : 5×4=20

(a) Let G and G′ be two groups and ϕ be a homomorphism from G onto G′. Prove that if G is commutative, then G′ is also commutative, and if G is cyclic then G′ is also cyclic.

Or

Show that any infinite cyclic group is isomorphic to the additive group of integers, and any finite cyclic group of order n is isomorphic to $\mathbb{Z}_n$, the group of integers modulo n.

(b) If in a ring R with unity $(xy)^2 = x^2y^2$, for all $x, y \in R$, then show that R is commutative.

Or

Let R be a finite (non-zero) integral domain. Then prove that $o(R) = p^n$, where p is a prime.

(c) Let G be a group. Show that the set of all inner automorphisms of G is a subgroup of the group of all automorphisms of G.

(d) Let R[x] be the ring of polynomials over a ring R. Show that R is commutative if and only if R[x] is commutative.

4. Answer the following questions : 10×4=40

(a) Prove that the set A_n of all even permutations of S_n $(n \geq 2)$ is a normal subgroup of S_n and $o(A_n) = \frac{1}{2} o(S_n)$. Find all the normal subgroups of S_4. 7+3=10

Or

Let f be a homomorphism from a group G onto a group G′. Let H be a subgroup of G and H′ be a subgroup of G′.

Prove that

(i) f(H) is a subgroup of G′.

(ii) $f^{-1}(H')$ is a subgroup of G containing K = Ker f.

(iii) there exists a one-to-one correspondence between the set of subgroups of G containing K and the set of subgroups of G'. 3+3+4=10

(b) Let R be a commutative ring. Prove that an ideal P of R is prime if and only if R/P is an integral domain. Moreover, if R is with unity and M is a maximal ideal of R such that $M^2 = \{o\}$, then show that for any other maximal ideal N of R, N = M. 6+4=10

Or

Show that the union of two subspaces of a vector space may not be a subspace. Consider the vector space $V(F) = F^2(F)$, where F is a field. Let $W_1 = \{(a, o) : a \in F\}$ and $W_2 = \{(o, b) : b \in F\}$ Show that $V = W_1 \oplus W_2$. 4+6=10

(c) Let G be a finite group and x, y be conjugate elements of G. Show that the number of distinct elements $g \in G$ such that $g^{-1}xg = y$ is $o(N(x))$. 10

Or

Prove that the number of elements of the conjugacy class c(a) of a group G of finite order is $o(G)/o(N(a))$, where N(a) is the normalizer of a. If Z(G) denotes the centre of G, then prove that

$$o(G) = o(Z(G)) + \sum_{a Z(G)} \frac{o(G)}{o(N(G))}. \qquad 10$$

(d) If D_1 and D_2 are two isomorphic integral domains, then show that their respective fields of quotients F_1 and F_2 are also isomorphic. 10

Or

Show that an integral domain can be imbedded into a field. 10