Total No. of printed pages = 7

3 (Sem 1) MAT M2

2015

MATHEMATICS

(Major)

Paper : 1.2

(Calculus)

Full Marks – 80

Time – Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following : $1 \times 10 = 10$

(a) Write down the nth derivative of $\log_e(ax+b)$.

(b) If $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$, find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

(c) Find $\frac{ds}{dx}$ for the curve $y^2 = 4ax$.

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(d) Find the radius of curvature at any point (s, Ψ) on the curve $s = c \log \sec \Psi$.

(e) Write down the asymptotes of the curve $x^2 - y^2 = a^2$.

(f) If $f(x, y) = x \cos y + y \cos x$, find f_{xy}.

(g) Choose the correct answer : $\int \frac{dx}{a^2 - x^2}$ equals

(i) $\frac{1}{a} \tan^{-1} \frac{x}{a} + c,\ a \neq 0$

(ii) $\frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c,\ |x| \neq |a|$

(iii) $\frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c,\ |x| \neq |a|$

(iv) $\sin^{-1} \frac{x}{a} + c,\ |x| < |a|$

(h) Write down the value of $\int_{-1}^{1} x|x| dx$

(i) Evaluate $\int_{0}^{\frac{\pi}{2}} \cos^4 x \, dx$

(j) Write down the intrinsic equation of the curve

$$y = a \log \sec\left(\frac{x}{a}\right),$$ the fixed point being the origin.

2. Answer the following questions : $2\times5=10$

(a) If $y = \cos^3 x$, find y_n

(b) Find $\frac{ds}{d\theta}$ for the curve $r = a(1+\cos\theta)$

(c) Prove that $\int_{0}^{\frac{\pi}{2}} \sin 2x \log \tan x \, dx = 0$

(d) Find the area of a hyperbola $xy = c^2$ bounded by the x-axis and the ordinates $x = a$ and $x = b$.

(e) Find the volume of solid generated by revolving about the x-axis, the area bounded by $y = \sin x$; $x = 0$, $x = \pi$.

3. Answer the following questions : $5\times4=20$

(a) If $u = \log(x^3+y^3+z^3-3xyz)$, show that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{3}{(x+y+z)^2}$$

Or

If $F(v^2-x^2, v^2-y^2, v^2-z^2) = 0$, where v is a function x, y, z, show that

$$\frac{1}{x}\frac{\partial v}{\partial x}+\frac{1}{y}\frac{\partial v}{\partial y}+\frac{1}{z}\frac{\partial v}{\partial z}=\frac{1}{v}$$

(b) Trace the curve

$$y^2 = x^2\,\frac{a+x}{a-x}$$

Or

Show that the portion of the tangent at any point on the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, intercepted between the axes is of constant lengths.

(c) Evaluate $\int \frac{\tan x}{\sqrt{a + b\tan^2 x}}\, dx,\ b > a$

Or

$$\int \frac{dx}{\sqrt{(x-\alpha)(\beta - x)}},\ \beta > \alpha$$

(d) Show that the area bounded by the parabola $\sqrt{x} + \sqrt{y} = \sqrt{a}$ and the coordinate axes is $\frac{1}{6}a^2$.

4. Answer either (a) or (b) :

(a) (i) If $u = \sin ax + \cos ax$, show that

$$u_n = a^n [1 + (-1)^n \sin 2ax]^{\frac{1}{2}}$$ 5

(ii) If $y = a\cos(\log x) + b\sin(\log x)$, prove that

$$x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$$ 5

(b) (i) State and prove the Euler's theorem on Homogeneous functions for two variables. 2+4=6

(ii) If $u = \phi(H_n)$, where H_n is a homogeneous function of degree n in x, y, z, then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = n\frac{F(u)}{F'(u)}$$

where $F(u) = H_n$. 4

5. Answer either (a) or (b) :

(a) (i) Find the asymptotes of the curve

$y^3 + x^2y + 2xy^2 - y + 1 = 0.$ 5

(ii) If ρ_1 and ρ_2 be the radii of curvature at the ends of a focal chord of the parabola $y^2 = 4ax$, then show that

$$\rho_1^{-\frac{2}{3}} + \rho_2^{-\frac{2}{3}} = (2a)^{-\frac{2}{3}}$$ 5

(b) Define double points and double cusp. Search for double points on the curve $x^4 - 2y^3 - 3y^2 - 2x^2 + 1 = 0$. 2+8=10

6. (a) If $u_n = \int_0^1 x^n \tan^{-1} x \, dx$, then prove that

$$(n+1)u_n + (n-1)u_{n-2} = \frac{\pi}{2} - \frac{1}{n}$$ 5

(b) If $J_n = \int \sec^n x \, dx$, then show that

$$(n-1)J_n = \tan x \sec^{n-2} x + (n-2)J_{n-2}$$ 5

7. (a) Find the total length of the astroid : 5

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

(b) Find the volume of the solid generated by revolving the cardioide $r = a(1 - \cos\theta)$ about the initial line. 5