

3 (Sem-3) MAT M 2

2014

MATHEMATICS

(Major)

Paper : 3.2

Full Marks : 80

Time : 3 hours

*The figures in the margin indicate full marks
for the questions*

GROUP—A

(**Linear Algebra**)

(Marks : 40)

1. Answer the following as directed : 1×6=6

(a) Let W be a subset of the vector space $\mathbb{R}^3(\mathbb{R})$ defined by

$$W = \{(x, y, z) \mid x, y, z \in \mathbb{R} \text{ and } x + 2y + 4z = 4\}$$

Is W a subspace? Justify.

(b) Show that each non-zero singleton set $\{x\}$ of a vector space is linearly independent.

(c) If M is the vector space of all $m \times n$ matrices, determine $\dim M$.

3. Answer any one part :

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- (a) (i) Let U and W be subspaces of a vector space $V(F)$. Then show that $V = U \oplus W \Leftrightarrow V = U + W$ and $U \cap W = \{0\}$.
- (ii) Show that the vectors $(1, 2, 5)$, $(2, 5, 1)$ and $(1, 5, 2)$ are linearly independent in the vector space $\mathbb{R}^3(\mathbb{R})$.
- (iii) If V is a finite dimensional vector space and $\{v_1, v_2, \dots, v_n\}$ is a linearly independent subset of V , then prove that it can be extended to form a basis of V . 4+2+4=10
- (b) (i) If W_1 and W_2 are subspaces of a finite dimensional vector space $V(F)$, then show that
- $$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$
- (ii) Define basis of a vector space. Determine whether or not the vectors $(1, 1, 2)$, $(1, 2, 5)$, $(5, 3, 4)$ form a basis of the vector space $\mathbb{R}^3(\mathbb{R})$. 5+5=10

4. Answer any two parts :

5×2=10

- (a) Define rank and nullity of a linear transformation T from a vector space $V(F)$ to a vector space $W(F)$. Find rank and nullity of the linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by

$$T(x, y, z) = (x + y + z, 2x + 2y + 2z) \quad 1+4=5$$

- (b) Let T be a linear operator on a vector space $V(F)$ and $\text{rank } T^2 = \text{rank } T$. Then show that $\text{range } T \cap \ker T = \{0\}$.

5

- (c) Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a linear mapping defined by

$$T(x, y, z) = (3x + 2y - 4z, x - 5y + 3z)$$

Find the matrix A representing T relative to the bases $B = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$ of $\mathbb{R}^3(\mathbb{R})$ and $B' = \{(1, 3), (2, 5)\}$ of $\mathbb{R}^2(\mathbb{R})$.

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5. Answer any one part :

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- (a) (i) Find the eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$$

- (ii) State the Cayley-Hamilton theorem.
Verify it for the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{pmatrix}$$

Hence find A^{-1} .

4+6=10

- (b) (i) Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Show that A and B have different characteristic polynomials but have the same minimal polynomial.

- (ii) Show that the system of equations

$$3x + y + z = 8$$

$$-x + y - 2z = -5$$

$$2x + 2y + 2z = 12$$

$$-2x + 2y - 3z = -7$$

is consistent and hence solve them.

5+5=10

GROUP—B

(Vector)

(Marks : 40)

6. Answer the following as directed : 1×4=4

(a) Evaluate $(\vec{r} \cdot \hat{i})\hat{i} + (\vec{r} \cdot \hat{j})\hat{j} + (\vec{r} \cdot \hat{k})\hat{k}$.

(b) Write the value of $\text{div}(\nabla\phi \times \nabla\psi)$.(c) If ϕ is a continuously differentiable scalar point function, the value of $\text{curl grad } \phi$ is

(i) 1

(ii) -1

(iii) $\vec{0}$

(iv) None of the above

(Choose the correct answer)

(d) If C is a closed curve, find $\oint_C \vec{r} \cdot d\vec{r}$.7. Answer the following : 2×3=6

(a) Prove the identity

$$\vec{a} \times [\vec{a} \times (\vec{a} \times \vec{b})] = (\vec{a} \cdot \vec{a})(\vec{b} \times \vec{a})$$

(b) Find $\text{div curl } \vec{F}$, if $\vec{F} = x^2y\hat{i} + xz\hat{j} + 2yz\hat{k}$.

(c) Interpret the relations

$$\vec{r} \cdot \frac{d\vec{r}}{dt} = 0 \quad \text{and} \quad \vec{r} \times \frac{d\vec{r}}{dt} = \vec{0}$$

8. Answer any one part :

10

(a) (i) Prove that

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) + (\vec{b} \times \vec{c}) \cdot (\vec{a} \times \vec{d}) + (\vec{c} \times \vec{a}) \cdot (\vec{b} \times \vec{d}) = 0$$

(ii) Prove that $\nabla^2\left(\frac{1}{r}\right) = 0$, where

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}.$$

(iii) If

$$\vec{A} = xyz\hat{i} + xz^2\hat{j} - y^3\hat{k}$$

and

$$\vec{B} = x^3\hat{i} - xyz\hat{j} + x^2z\hat{k}$$

calculate

$$\frac{\partial^2 \vec{A}}{\partial y^2} \times \frac{\partial^2 \vec{A}}{\partial x^2}$$

at the point (1, 1, 3).

3+4+3=10

(b) (i) Prove that

$$[\vec{b} + \vec{c}, \vec{c} + \vec{a}, \vec{a} + \vec{b}] = 2[\vec{a}, \vec{b}, \vec{c}]$$

Hence show that $\vec{a}, \vec{b}, \vec{c}$ are coplanar, iff $\vec{b} + \vec{c}, \vec{c} + \vec{a}, \vec{a} + \vec{b}$ are coplanar.

(ii) Prove that

$$\text{curl}(f\vec{F}) = \text{grad } f \times \vec{F} + f(\text{curl } \vec{F})$$

where f is a scalar point function.

(iii) Find the unit normal vector to the

surface $x^2y + 2xz = 4$ at the point

(2, -2, 3).

$$4+3+3=10$$

9. Answer any two parts :

$$5 \times 2 = 10$$

(a) If \vec{a} , \vec{b} , \vec{c} are three vectors, then prove that

$$(\vec{a} \vec{b} \vec{c})^2 = \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} & \vec{a} \cdot \vec{c} \\ \vec{b} \cdot \vec{a} & \vec{b} \cdot \vec{b} & \vec{b} \cdot \vec{c} \\ \vec{c} \cdot \vec{a} & \vec{c} \cdot \vec{b} & \vec{c} \cdot \vec{c} \end{vmatrix} \quad 5$$

(b) Prove that

$$\text{curl}(\vec{a} \times \vec{b}) = (\vec{b} \cdot \nabla)\vec{a} - (\vec{a} \cdot \nabla)\vec{b} + \vec{a} \text{div } \vec{b} - \vec{b} \text{div } \vec{a} \quad 5$$

(c) Show that

$$\int \left(\vec{r} \times \frac{d^2 \vec{r}}{dt^2} \right) dt = \vec{r} \times \frac{d\vec{r}}{dt} + C$$

where C is an arbitrary constant vector.If $\vec{r}(t) = 5t^2 \hat{i} + t\hat{j} - t^3 \hat{k}$, then prove that

$$\int_1^2 \left(\vec{r} \times \frac{d^2 \vec{r}}{dt^2} \right) dt = -14\hat{i} + 75\hat{j} - 15\hat{k} \quad 2+3=5$$

10. Answer any one part :

10

(a) (i) Evaluate $\int_V (2x+y) dV$, where V is

a closed region bounded by the cylinder $z=4-x^2$ and the planes $x=0$, $y=0$, $y=2$ and $z=0$.

(ii) Verify Green's theorem in the plane for

$$\oint_C (x^2 - xy^3) dx + (y^2 - 2xy) dy$$

where C is the square with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$ and $(0, 2)$.

5+5=10

(b) (i) Find the work done in moving a particle once around a circle C in the xy -plane, if the circle has centre at the origin and radius 2 and, if the force field \vec{F} is given by

$$\vec{F} = (2x - y + 2z)\hat{i} + (x + y - z)\hat{j} + (3x - 2y - 5z)\hat{k}$$

(ii) State Gauss' divergence theorem.

Using it, evaluate $\iint_S \vec{F} \cdot \hat{n}$, where

$\vec{F} = x\hat{i} - y\hat{j} + (z^2 - 1)\hat{k}$ and S is the cylinder formed by the surfaces $z=0$, $z=1$, $x^2 + y^2 = 4$.

5+5=10
