

2014

MATHEMATICS

( Major )

Paper : 3.1

( Abstract Algebra )

Full Marks : 80

Time : 3 hours

*The figures in the margin indicate full marks  
for the questions*

1. Answer the following as directed :  $1 \times 10 = 10$

(a) Let  $G$  and  $G'$  be finite groups such that  $\gcd(o(G), o(G')) = 1$ . Define a homomorphism from  $G$  to  $G'$ .

(b) State the fundamental theorem of group homomorphism.

(c) Let  $f : G \rightarrow G'$  be a group homomorphism. Let  $a \in G$  be such that  $o(a) = n$  and  $o(f(a)) = m$ . Then  $o(f(a)) / o(a)$  and  $f$  is one-one if and only if

(i)  $m > n$

(ii)  $m < n$

(iii)  $m = n$

(iv)  $m = n = 1$

(Choose the correct option)

(d) Let  $R = \{0, 1, 2\} \text{ mod } 3$ . What is the characteristic of  $R$ ?

(e) State whether True or False :

If an element  $a$  of a group  $G$  has only two conjugates in  $G$ , then  $N(a)$  is a normal subgroup of  $G$ .

( $N(a)$ : normalizer of  $a$  in  $G$ )

(f) State Cauchy's theorem for a finite group  $G$ .

(g) If  $T$  is an automorphism of a group  $G$ , then  $o(Ta) = o(a)$  for all  $a \in G$ . Now, for all  $a, b \in G$

(i)  $o(bab^{-1}) = o(b)$

(ii)  $o(bab^{-1}) = o(a)$

(iii)  $o(bab^{-1}) = o(Tb)$

(iv)  $o(bab^{-1}) = 2$

(Choose the correct option)

(h) Give an example of a Euclidean domain.

- (i) Let  $R[x]$  be the ring of polynomials of a ring  $R$  and let

$$f(x) = a_0 + a_1x + \cdots + a_mx^m$$

$$\text{and } g(x) = b_0 + b_1x + \cdots + b_nx^n$$

If  $f(x) + g(x) \neq 0$ , then

(i)  $\deg(f(x) + g(x)) \leq \max(m, n)$

(ii)  $\deg(f(x) + g(x)) \geq m + n$

(iii)  $\deg(f(x) + g(x)) = m + n$

(iv)  $\deg(f(x) + g(x)) \geq \max(m, n)$

(Choose the correct option)

- (j) State True or False :

In a principal ideal domain, every non-zero prime ideal is maximal.

2. Answer the following questions : 2×5=10

(a) Let  $\mathbb{Z}$  be the additive group of integers and  $\phi: \mathbb{Z} \rightarrow \mathbb{Z}$  be defined by  $\phi(x) = x + 1$ ,  $x \in \mathbb{Z}$ . Examine if  $\phi$  is a homomorphism.

(b) If  $R$  is a ring with no non-zero nilpotent elements, then show that for any idempotent  $e$ ,  $ex = xe \forall x \in R$ .

(c) The sum of two subspaces of a vector space is again a subspace.

Justify whether it is true or false.

- (d) Let  $G$  be a group and  $Z(G)$  be the centre of  $G$ . Show that if  $\text{cl}(a) = \{a\}$ , then  $a \in Z(G)$ . ( $\text{cl}(a)$  : the conjugacy class of  $a$ )
- (e) Let  $f$  be a homomorphism from a ring  $R$  onto a ring  $R'$ . If  $e$  is the unity of  $R$ , then  $f(e)$  is the unity of  $R'$ .

Justify whether this statement is true or false.

3. Answer the following questions : 5×4=20

- (a) Let  $G = (\mathbb{R}, +)$ ,  $G' = (\{Z \in \mathbb{C} : |Z|=1\}, \cdot)$  and  $\phi : G \rightarrow G'$  is defined by

$$\phi(x) = \cos 2\pi x + i \sin 2\pi x, \quad x \in \mathbb{R}$$

Prove that  $\phi$  is a homomorphism and determine  $\ker \phi$ .

Or

If  $f : G \xrightarrow{\text{onto}} G'$  is a group homomorphism, prove that  $H$  is a normal subgroup of  $G$  if and only if  $f(H)$  is a normal subgroup of  $G'$ .

- (b) If  $R$  is a division ring, then show that the centre  $Z(R)$  of  $R$  is a field.

Or

Let  $R$  be a ring having more than one element such that  $aR = R$ , for all  $0 \neq a \in R$ . Show that  $R$  is a division ring.

(c) Prove that a group of order  $p^2$ , where  $p$  is prime, is Abelian.

(d) Prove that every ideal in a Euclidean domain is a principal ideal.

4. Answer the following questions : 10×4=40

(a) Let  $H$  and  $K$  be two normal subgroups of a group  $G$  such that  $H \subseteq K$ . Prove that

$$\frac{G}{K} \cong \frac{G/H}{K/H} \quad 10$$

Or

Let  $G$  be the additive group of reals and  $N$  be the subgroup of  $G$  consisting of integers. Prove that  $\frac{G}{N}$  is isomorphic to the group  $H$  of all complex numbers of absolute value 1 under multiplication.

(b) Let  $A, B, C$  be ideals of a ring  $R$  such that  $B \subseteq A$ . Show that

$$A \cap (B + C) = (A \cap B) + (A \cap C) = B + (A \cap C) \quad 10$$

Or

Let  $R$  be a commutative ring with unity. Show that every maximal ideal of  $R$  is also a prime ideal. Moreover, prove that if every ideal of  $R$  is prime, then  $R$  is a field. 3+7=10

- (c) State Sylow's 1st and 3rd theorems for a group  $G$ . Let  $o(G) = pq$ , where  $p, q$  are distinct primes such that  $p < q, p \nmid q-1$ . Show that  $G$  is cyclic. 2+8=10

Or

Let  $G$  be a finite group and  $a \in G$ . Prove that

$$o(\text{cl}(a)) = \frac{o(G)}{o(N(a))} \quad 10$$

- (d) Show that  $\mathbb{Z}[\sqrt{-2}] = \{a + \sqrt{-2}b : a, b \in \mathbb{Z}\}$  is a Euclidean domain. 10

Or

Prove that any ring can be imbedded into a ring with unity.

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