2013

MATHEMATICS

(Major)

Paper : 3.1

(Abstract Algebra)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer/Choose the correct one from the following : 1×10=10

(a) Define a homomorphism from the group of integers to the group of even integers.

(b) A homomorphism f from a group G into a group G' is a monomorphism if and only if

(i) $\ker f = \phi$

(ii) $\ker f = \{e\}$, where e is the identity element of G

(iii) $\ker f$ is a normal subgroup of G

(iv) $o(\ker f) = 2$

(c) Find the order of the permutation of the following :

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 5 & 1 & 3 \end{pmatrix}$$

(d) Give an example of a division ring which is not a field.

(e) Define inner automorphism of a group G.

(f) State Sylow's third theorem for a group G.

(g) Let T be an automorphism of a group G. Then $\forall\, a \in G$

(i) $o(Ta) = o(G)$

(ii) $o(Ta) = o(a)$

(iii) $o(Ta) = 1$

(iv) $o(Ta) = 2$

(h) Let T be a ring of polynomials over a ring R. If R has unity 1, then what will be the unity of T?

(i) Define the field of quotients of an integral domain.

(j) State whether the following statement is True or False :

A field F is always a principal ideal domain.

2. Answer the following questions : $2\times5=10$

(a) If $f: G_1 \to G_2$ is a homomorphism for two groups G_1 and G_2, then prove that $\ker f$ is a normal subgroup of G_1.

(b) If D is an integral domain and if $na = 0$ for some $0 \neq a \in D$ and some integer $n \neq 0$, then show that the characteristic of D is finite.

(c) Consider the vector space $\mathbb{R}^2$ over $\mathbb{R}$. Show that $W_1 = \{(a, 0) : a \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$.

(d) If $f: G \to G$ be such that $f(a) = a^n$, (where n is a positive integer) is an automorphism, then prove that

$$a^{n-1} \in Z(G) \ \forall\, a \in G$$

where $Z(G)$ is the centre of the group G.

(e) Let R be a ring with unity 1 and f is a homomorphism of R into an integral domain R'. If $\ker f \neq R$, prove that $f(1)$ is the unity of R'.

3. Answer the following questions : $5 \times 4 = 20$

(a) Let f be a homomorphism from a group G_1 to a group G_2. Let $a \in G_1$ be such that $o(a) = n$ and $o(f(a)) = m$. Show that—

(i) $o(f(a)) / o(a)$;

(ii) f is one-one iff $m = n$.

Or

If H is any subgroup of the permutation group S_n $(n \geq 2)$, then prove that either all permutations in H are even or exactly half are even.

(b) If in a ring R, $x^2 = x$ $\forall\, x \in R$, then show that R is commutative.

Or

Let R be a ring with unity such that R has no right ideals except $\{0\}$ and R. Show that R is a division ring.

(c) Let G be a finite Abelian group of order n, where n is an odd number (> 1). Show that G has a non-trivial automorphism.

(d) Let f be a homomorphism from a ring R_1 to a ring R_2. If A is an ideal of R_1, prove that $f(A)$ is an ideal of $f(R_1)$.

4. Answer the following questions : $10 \times 4 = 40$

(a) State and prove Cayley's theorem on a group G. 10

Or

Let G and G' be two groups and $f : G \to G'$ be an onto homomorphism with $\ker f = K$. Let H' be a subgroup of G'. Define

$$H = \{x \in G : f(x) \in H'\}$$

Also prove that—

(i) H is a normal subgroup of G if and only if H' is a normal subgroup of G';

(ii) if H' is normal in G', then

$$\frac{G'}{H'} \cong \frac{G}{H}$$ 4+6=10

(b) If in a ring R, the equation $ax = b (a \neq 0)$ has a solution, then show that R is a division ring. Also prove that the centre $Z(R)$ of a division ring R is a field. 6+4=10

Or

Let V be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$. Let

$$V_e = \{f \in V : f \text{ is even}\} \text{ and}$$
$$V_o = \{f \in V : f \text{ is odd}\}$$

Show that V_e and V_o are subspaces of V and $V = V_e \oplus V_o$. 10

(c) Let G be a finite group and $k(G)$ = number of conjugate classes of G is 3. Show that either G is a cyclic group of order 3 or is a non-Abelian group of order 6. 10

Or

Let G be a finite group and let p be a prime such that $p / o(G)$. Prove that there exists $x \in G$ such that $o(x) = p$.

(d) Show that any ring R with unity can be imbedded into a ring of endomorphisms of some additive Abelian group. 10

Or

Define Euclidean domain. Show that the ring of integers $\mathbb{Z}$ is a Euclidean domain. Prove that every ideal in a Euclidean domain is a principal ideal. 2+4+4=10

★ ★ ★