# 2013

## MATHEMATICS

( Major )

Paper : 1.2

**( Calculus )**

*Full Marks* : 80

*Time* : 3 hours

*The figures in the margin indicate full marks for the questions*

1. Answer the following questions : 1×10=10

(a) Write down the $n$th derivative of

$$\frac{1}{x+1}$$

(b) If $z = x^2 y^3 \tan^{-1}\frac{y}{x}$, obtain

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$

(c) Find the polar subnormal to the curve $r = a(1-\cos\theta)$ at $\theta = \frac{\pi}{2}$.

(d) Obtain the radius of curvature at the point $(s, \psi)$ of the curve $s = c\tan\psi$.

(e) Write down the maximum number of asymptotes that the curve

$$x^2y^2 = a^2(x^2 + y^2)$$

may have.

(f) If

$$f(x, y) = \sin^{-1}\frac{y}{x}$$

find $\frac{\partial f}{\partial x}$.

(g) Obtain :

$$\int \frac{dx}{(1+x)\sqrt{x}}$$

(h) Write down the value of

$$\int_{-a}^{a} \frac{xe^{x^2}}{1+x^2}\,dx$$

(i) Evaluate :

$$\int_0^{\pi/2} \cos^6 x\,dx$$

(j) Write down the area bounded by the curve given by $x = a\cos\theta$, $y = b\sin\theta$; $0 \le \theta \le 2\pi$.

**2.** Answer the following questions : $2\times5=10$

(a) If $y = \cos 2x \cos x$, find $y_n$.

(b) Show that in the equiangular spiral $r = ae^{\theta \cot \alpha}$, the tangent is inclined at a constant angle to the radius vector.

(c) Prove that

$$\int_0^{\pi} x \cos^4 x\, dx = \frac{3\pi^2}{16}$$

(d) Determine the length of an arc of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, measured from the vertex (0, 0).

(e) Show that the area bounded by the cardioide $r = a(1 + \cos\theta)$ is $\frac{3}{2}\pi a^2$.

**3.** Answer the following questions : $5\times4=20$

(a) If $u = f(x, y)$, where $x = r\cos\theta$, $y = r\sin\theta$, show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

(b) Trace the curve

$$y^2(a^2 + x^2) = x^2(a^2 - x^2)$$

*Or*

Show that the portion of the tangent at any point on the curve

$$x^{2/3} + y^{2/3} = a^{2/3}$$

intercepted between the axes is of constant length.

(c) Integrate :

$$\int \frac{dx}{\cos(x-\alpha)\cos(x-\beta)}$$

*Or*

$$\int \frac{dx}{(2x+3)\sqrt{x^2+3x+2}}$$

(d) The circle $x^2 + y^2 = a^2$ revolves round the $x$-axis. Find the surface area of the solid generated.

4. Answer *either (a) or (b)* : 10

(a) (i) State and prove Leibnitz theorem. 1+4

(ii) If $\log y = \tan^{-1} x$, prove that

$$(1+x^2)y_{n+2} + (2nx + 2x - 1)y_{n+1} + n(n+1)y_n = 0$$ 5

(b) (i) If $f(x, y, z)$ be a homogeneous function in $x, y, z$ of degree $n$, having continuous partial derivatives, show that

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=nf$$ 5

(ii) If

$$u = x\phi\left(\frac{y}{x}\right)+\psi\left(\frac{y}{x}\right)$$

show that

$$x^2U_{xx}+2xyU_{xy}+y^2U_{yy}=0$$ 5

5. Answer the following : 10×3=30

(a) Define curvature and radius of curvature for a plane curve. Prove that the radius of curvature for the curve $y = f(x)$ at the point $P(x, y)$ is given by

$$\rho=\frac{(1+y_1^2)^{3/2}}{y_2}$$

where $y_1=\frac{dy}{dx}$, $y_2=\frac{d^2y}{dx^2}$ and $y_2 \neq 0$.

If $\rho_1$ and $\rho_2$ be the radii of curvature at the ends of a focal chord of the parabola $y^2=4ax$, show that

$$\rho_1^{-2/3}+\rho_2^{-2/3}=(2a)^{-2/3}$$ 2+4+4

*Or*

Define double points. Show that the necessary and sufficient conditions for any point $P(x, y)$ on the curve $f(x, y) = 0$ to be a multiple point are $f_x(x, y) = 0$ and $f_y(x, y) = 0$. Search for double points on the curve

$$x^4 + y^3 + 2x^2 + 3y^2 = 0$$ 1+4+5

(b) (i) If

$$I_n = \int \tan^n x\,dx, \ (n > 1)$$

prove that

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

and hence obtain

$$\int \tan^3 x\,dx$$ 5

(ii) If

$$U_n = \int_0^{\pi/2} x^n \sin x\,dx, \ (n \geq 1)$$

show that

$$U_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)U_{n-2}$$

and hence derive the value of

$$\int_0^{\pi/2} x \sin x\,dx$$ 5

(c) (i) Find the total length of the astroid

$$x^{2/3} + y^{2/3} = a^{2/3}$$ 5

(ii) The curve

$$y^2(2a - x) = x^3$$

revolves round its asymptote. Show that the volume of the solid generated is $2\pi^2 a^3$. 5

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