3 (Sem-1) MAT M 1

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MATHEMATICS

(Major)

Paper : 1.1

(Algebra and Trigonometry)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

(Objective-type Questions)

(*Marks* : 10)

1. Answer/Choose the correct option : 1×10=10

(a) The set $\pi = \{\{1, 2, 3\}, \{4, 5\}, \{6\}\}$ is a partition of the set

(i) $X_1 = \{\{1, 2\}\}$

(ii) $X_2 = \{\{1, 2, 3, 4\}\}$

(iii) $X_3 = \{\{1, 2, 3, 4, 5, 6\}\}$

(iv) $X_4 = \{\{4, 5\}, \{6\}\}$

(b) Determine if the binary operation $\circ$ on $\mathbb{R}$ commutative in each of the following cases :

(i) $x \circ y = x^2 y$

(ii) $x \circ y = \min(x, y)$

(iii) $x \circ y = x^y + y^x$

(iv) $x \circ y = 1$

(c) Let $f : R \to R$ and $g : R \to R$ be any two maps and define $f \circ g(x) = g(f(x))$, $f(x) = \sin x$, $g(x) = x^2$. Determine

(i) $(g \circ f)(x)$ (ii) $f(x^2)$

(iii) $f(x) = x + 1$ (iv) $(g \circ f)(x)$

(d) If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then A^n is

(i) $\begin{bmatrix} \sin n\alpha & \cos n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$

(ii) $\begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$

(iii) $\begin{bmatrix} -\sin n\alpha & \cos n\alpha \\ \cos n\alpha & \sin n\alpha \end{bmatrix}$

(iv) $\begin{bmatrix} -\sin n\alpha & -\cos n\alpha \\ \cos n\alpha & -\sin n\alpha \end{bmatrix}$

(e) The product of eigenvalues of a matrix A is equal to

(i) the sum of the eigenvalues of its matrix

(ii) the sum of the elements of the principal diagonal

(iii) its determinant

(iv) the reciprocal of the eigenvalues

(f) A real quadratic form $X'AX$ in n variables is said to be positive definite

(i) if all the eigenvalues of $A < 0$

(ii) if all the eigenvalues of $A \geq 0$ and at least one eigenvalue $= 0$

(iii) if some of the eigenvalues of A are positive and others negative

(iv) if all the eigenvalues of $A > 0$

(g) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are latent roots of a matrix A, then A^2 has latent roots

(i) $2\lambda_1, 2\lambda_2, \ldots, 2\lambda_n$

(ii) $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n}$

(iii) $\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2$

(iv) $\frac{1}{2}\lambda_1, \frac{1}{2}\lambda_2, \ldots, \frac{1}{2}\lambda_n$

(h) Let the ranks of the coefficient matrix A and the augmented matrix K, by reducing A to the triangular form by elementary row operations are obtained as r and r'. Then

(i) the system becomes consistent if $r \neq r'$

(ii) the system becomes consistent and there is a unique solution if $r = r' = n$

(iii) the system is consistent and there are infinite number of solutions if $r = r' = n$

(iv) the equations have $(n-r)$ a system of linearly dependent solutions if $r < n$

(i) The rank of the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{pmatrix}$$

is

(i) 1 (ii) 2

(iii) 3 (iv) 0

(j) If α, β, γ are the roots of the equation $x^3 + px + q = 0$, then the value of $\sum \alpha^3 \beta$ is

(i) $2p^2$

(ii) $-2p^2$

(iii) $3p^3$

(iv) $-3p^3$

SECTION—B

(Very short-type Questions)

(*Marks* : 10)

2. Give brief answers to the following questions : 2×5=10

(a) Show that the relation defined on $\mathbb{N} \times \mathbb{N}$ by $(a, b) \sim (c, d)$ iff $a + d = b + c$ is an equivalence relation.

(*b*) If $f : A \to B$, $g : B \to C$ and $h : C \to D$, then examine whether $h(gf)$ and $(hg)f$ are equal or not.

(*c*) Solve the equation $x^3 + 6x + 20 = 0$ if one root is $1 + 3i$.

(*d*) Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

(*e*) If $\alpha + i\beta = 1/(a + ib)$, then prove that

$$(\alpha^2 + \beta^2)(a^2 + b^2) = 1$$

SECTION—C

(Short answer-type Questions)

(*Marks* : 20)

3. Answer any *four* parts : 5×4=20

(*a*) Prove that an injective mapping from a finite set to itself is bijective.

(*b*) Show that if a, b are arbitrary elements of a group G, then $(ab)^2 = a^2b^2$ if and only if G is Abelian.

(c) Solve the equation

$$x^4 - 2x^3 - 21x^2 + 22x + 40 = 0$$

whose roots are in AP.

(d) Test the consistency and solve

$$5x + 3y + 7z = 4$$
$$3x + 26y + 2z = 9$$
$$7x + 2y + 10z = 5$$

(e) If $\sin^{-1}(u + iv) = \alpha + i\beta$, then prove that $\sin^2\alpha$ and $\cosh^2\beta$ are the roots of the equation

$$x^2 - x(1 + u^2 + v^2) + u^2 = 0$$

SECTION—D

(*Marks* : 40)

4. Answer any *two* parts : 5×2=10

(a) Show that an equivalence relation ~ over a set induces a portion of the set. Is the converse of the above also true?

(b) If $f : A \to B$ and $g : B \to C$ be one-one and onto maps, then show that $g \circ f$ is inversible and

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

(c) Show that a finite non-empty semigroup satisfying the cancellation laws is a group.

(d) Show that the set of all n, nth roots of unity form a finite Abelian group of order n under ordinary multiplication as composition.

5. Answer any *two* parts : 5×2=10

(a) Prove that the order of a cyclic group is equal to the order of any generator of the group.

(b) Prove that every finite group G is isomorphic to a permutation group.

(c) If H, K are subgroups of a group G, then show that HK is a subgroup of G if and only if $HK = KH$.

(d) State and prove Lagrange's theorem on order of a finite group and its subgroups.

6. Answer any *two* parts : 5×2=10

(a) (i) Prove that

$$(1+\cos\theta+i\sin\theta)^n+(1+\cos\theta-i\sin\theta)^n$$
$$=2^{n+1}\cos^n(\theta/2)(\cos n\theta/2) \quad 3$$

(ii) Mark by points on the Argand diagram, all the values of $(1+i)^{1/5}$ and verify that they form a pentagon. 2

(b) (i) If

$$u=\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

then prove that

$$\theta=-i\log\tan\left(\frac{\pi}{4}+\frac{iu}{2}\right) \text{ and } \tanh\frac{u}{2}=\tan\frac{\theta}{2} \quad 3$$

(ii) Find the value of π using Gregory's series. 2

(c) (i) If A is an $n\times n$ square matrix, then show that $|\text{adj } A|=|A|^{n-1}$. 3

(ii) Find eigenvalue and eigenvectors of

$$\begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix} \quad 2$$

(d) (i) If $|A| \neq 0$, then show that

$$\operatorname{adj}(A^{-1}) = (\operatorname{adj} A)^{-1}$$ 2

(ii) Reduce to the normal form

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

of the matrix

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$ 3

7. Answer any *two* parts : 5×2=10

(a) Investigate for what values of λ and μ, the simultaneous equations

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

have *(i)* no solution, *(ii)* a unique solution and *(iii)* infinite number of solutions.

(b) Prove that every square matrix satisfies its own characteristic equation.

(c) Show that every square matrix can be uniquely expressed as a sum of a symmetric and a skew-symmetric matrix.

(d) Prove that the necessary and sufficient conditions for a square matrix A to possess an inverse is that $|A| \neq 0$.

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