2013

MATHEMATICS

(Major)

Paper : 3.2

(Linear Algebra and Vector)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

GROUP—A

(Linear Algebra)

(*Marks* : 40)

1. Answer the following : 1×6=6

   (a) If W is the vector space of all polynomials of degree $\leq n$, determine $\dim W$.

   (b) Let W be a subspace of the vector space $\mathbb{R}^3$ over $\mathbb{R}$. If $\dim W = 1$, what does W represent geometrically?

(c) Let $F:\mathbb{R}^2 \to \mathbb{R}^2$ be the mapping defined by $F(x, y) = (x+1, y+2)$. Is F linear? Justify.

(d) Let S and T be linear operators on $\mathbb{R}^2$ defined by $S(x, y) = (y, x)$ and $T(x, y) = (0, x)$ respectively. Find the formula that defines the operator ST.

(e) If A is a non-singular matrix, then write the relation between eigenvalues of A and eigenvalues of A^{-1}.

(f) Let $AX = 0$ be a homogeneous system of m equations in n unknowns and S be the solution space of the system. If rank $A = k$, then determine dimension of S.

2. Answer the following : 2×2=4

(a) Let V be the vector space $\mathbb{R}^3$ over $\mathbb{R}$. If U and W are subspaces of V, where U is the xy-plane and V is the yz-plane, then show that $V = U + W$. Is the sum direct? Justify.

(b) Find all eigenvalues of

$$A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$$

over the real field $\mathbb{R}$.

3. Answer any *one* part : 10

(a) (i) For any two subspaces W_1 and W_2 of a vector space V over a field F, prove that $W_1 + W_2$ is the subspace of V spanned by $W_1 \cup W_2$.

(ii) Let V be a vector space of dimension n over F. Show that any linearly independent set of n vectors of V is a basis of V.

(iii) Let U and W be distinct 4-dimensional subspaces of a vector space V of dimension 6. Find the possible dimension of $U \cap W$. 4+3+3

(b) (i) What do you mean by the term 'complement' of a vector space? Prove that every subspace of a finite dimensional vector space has a complement.

(ii) Let W be a subspace of a finite dimensional vector space V. Prove that

$$\dim \frac{V}{W} = \dim V - \dim W$$

5+5

4. Answer any *two* parts : 5×2=10

(a) Let U be a finite dimensional vector space and $T: U \to V$ be a linear transformation. Establish :

$$\dim U = \dim(\ker T) + \dim(\operatorname{Im} T)$$

(b) Let T be the linear operator on $\mathbb{R}^3$ defined by

$$T(x, y, z) = (2y + z,\ x - 4y,\ 3x)$$

Find the matrix of T relative to the basis

$$\{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$$

Also verify that the action of T is preserved by the matrix representation. Is T non-singular?

(c) Consider the following bases of $\mathbb{R}^2$ over $\mathbb{R}$:

$$\{e_1 = (1, 0),\ e_2 = (0, 1)\} \text{ and } \{f_1 = (1, 3),\ f_2 = (2, 5)\}$$

Find the transition matrix P from $\{e_i\}$ to $\{f_i\}$ and the transition matrix Q from $\{f_i\}$ to $\{e_i\}$. Hence show that $Q = P^{-1}$.

5. Answer any *one* part : 10

(a) (i) Let A and B be two $n \times n$ matrices. Show that AB and BA have same eigenvalues.

(ii) Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix}$$

Is A similar to a diagonal matrix? If so, find one such matrix.

(iii) Verify Cayley-Hamilton for the matrix

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

Hence find A^{-1}. 3+3+4

(b) (i) Prove that the minimum polynomial of a square matrix exists and is unique.

(ii) For what values of λ and μ the system of equations

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

has a unique solution? 5+5

GROUP—B

(Vector)

(*Marks* : 40)

6. Answer the following : 1×4=4

(a) If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors such that $\vec{a}$ and $\vec{b}$ are parallel vectors, then determine

$$\vec{a}\cdot(\vec{b}\times\vec{c})$$

(b) Evaluate

$$\hat{i}\cdot(\hat{j}\times\hat{k})+\hat{j}\cdot(\hat{k}\times\hat{i})+\hat{k}\cdot(\hat{i}\times\hat{j})$$

(c) Write the geometrical interpretation of grad ϕ.

(d) Find div $\vec{v}$, where

$$\vec{v}=3x^2y\hat{i}+z\hat{j}+x^2\hat{k}$$

7. Answer the following : 2×3=6

(a) Show that

$$\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})=2\vec{a}$$

(b) Find the total work done in moving a particle in a force field, given by

$$\vec{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$$

along the curve $x = t^2 + 1$, $y = 2t^2$, $z = t^3$ from $t = 0$ to $t = 1$.

(c) If S is a closed surface, $\hat{n}$ is the outward drawn normal to S and V is the volume enclosed by S, then evaluate

$$\iiint_V \text{div}\ \hat{n}\, dv$$

8. Answer any *one* part : 10

(a) (i) Find the equation of the plane passing through the points $A(2, -1, 1)$, $B(3, 2, -1)$ and $C(-1, 3, 2)$.

(ii) Evaluate

$$\nabla \cdot (\vec{A} \times \vec{r}) \text{ if } \nabla \times \vec{A} = \vec{0}$$

(iii) If $\vec{f} = xy\hat{i} - z\hat{j} + x^2\hat{k}$ and C is the curve $x = t^2$, $y = 2t$, $z = t^3$ from $t = 0$ to $t = 1$, then evaluate $\int_C \vec{f} \times d\vec{r}$.

3+3+4

(b) (i) Find the volume of the parallelopipe whose edges are represented by

$$\vec{A} = 2\hat{i} - 3\hat{j} + 4\hat{k}$$
$$\vec{B} = \hat{i} + 2\hat{j} - \hat{k}$$
$$\vec{C} = 3\hat{i} - \hat{j} + 2\hat{k}$$

(ii) Find the equation of the tangent plane to the surface

$$2xz^2 - 3xy - 4x = 7$$

at the point (1, −1, 2).

(iii) If $\vec{f} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$, then evaluate

$$\iiint_V \nabla \times \vec{f}\, dv$$

where V is bounded by the coordinate planes and the plane $2x + 2y + z = 4$. 3+3+4

9. Answer any *two* parts : 5×2=10

(a) Establish the following :

$$\nabla \cdot (\phi \vec{A}) = (\nabla \phi) \cdot \vec{A} + \phi (\nabla \cdot \vec{A})$$

Hence evaluate $\nabla \cdot (r^3 \vec{r})$.

(b) Find the constants a and b so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface, $4x^2y + z^3 = 4$ at the point $(1, -1, 2)$.

(c) If $\phi(x, y, z)$ is any solution of Laplace's equation, then show that $\nabla\phi$ is a vector which is both solenoidal and irrotational.

10. Answer any *one* part : 10

(a) (i) Prove

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$$

(ii) If $\vec{v} = \vec{w} \times \vec{r}$, prove that $\vec{w} = \frac{1}{2}\text{curl}\vec{v}$ where $\vec{w}$ is a constant vector.

(iii) Using Stokes theorem, evaluate $\oint_C \vec{f} \cdot d\vec{r}$, where

$$\vec{f} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$$

and C is the boundary of the triangle with vertices at (0, 0, 0), (1, 0, 0) and (1, 1, 0). 3+3+4

(b) (i) Show that any four vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ satisfy

$$[\vec{a}\,\vec{b}\,\vec{c}]\vec{d} = [\vec{b}\,\vec{c}\,\vec{d}]\vec{a} + [\vec{c}\,\vec{a}\,\vec{d}]\vec{b} + [\vec{a}\,\vec{b}\,\vec{d}]\vec{c}$$

(ii) Prove that a vector function, $\vec{f}(t)$ is of constant direction if and only if $\vec{f}$ and $\dfrac{d\vec{f}}{dt}$ are perpendicular to each other.

(iii) Evaluate

$$\iint_S \vec{F}\cdot\hat{n}\,dS$$

where $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ and S is the surface of the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$ and $z=1$. 3+3+4

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