3 (Sem–1) MAT M 2



2012

# MATHEMATICS

( Major )

Paper : 1.2

( **Calculus** )

*Full Marks* : 80

*Time* : 3 hours

*The figures in the margin indicate full marks for the questions*

1. Answer the following questions : 1×10=10

(a) Write down the *n*th derivative of $\log(x+1)$.

(b) What is the degree of the homogeneous function $f(x, y) = x^2y^3\phi(x/y) + x^5\psi(y/x)$?

(c) Write down the subtangent for the curve $\log y = x \log a$.

(d) Find the radius of curvature at any point $(s, \psi)$ on the curve $s = c\log\sec\psi$.

(e) Write the equation of the asymptote to the curve $xy - ax - by = 0$ parallel to $x$-axis.

(f) If $f(x, y) = \log(x^2 + y^2)$, find $\frac{\partial f}{\partial x}$.

(g) If $I = \int \frac{dx}{(1-x)\sqrt{x}}$, find $I$.

(h) What is the value of $\int_0^{\pi} \frac{\sin 4x}{\sin x} dx$?

(i) Evaluate :

$$\int_{-\pi/2}^{\pi/2} x^9 \cos x \, dx$$

(j) Find the complete perimeter of the curve

$$x = \frac{1-t^2}{1+t^2}, \; y = \frac{2t}{1+t^2}$$

**2.** Answer the following questions : $2 \times 5 = 10$

(a) If $y = \cos^3 x$, find $y_n$.

(b) Prove that the pedal equation of the curve $r = e^{\theta}$ is $2p^2 = r^2$.

(c) If $I_n = \int_0^{\pi/2} \sin^n x \, dx$, prove that

$$I_n = \frac{n-1}{n} I_{n-2}, \; n > 1$$

(d) Prove that the length of the arc of the curve given by $x = e^{\theta} \sin\theta$, $y = e^{\theta} \cos\theta$ between the points $\theta = 0$ and $\theta = \frac{\pi}{2}$ is $\sqrt{2}(e^{\pi/2} - 1)$.

(e) Show that the area enclosed by the curve $r = a\cos 2\theta$ is $\frac{1}{2}\pi a^2$.

**3.** Answer the following questions : 5×4=20

(a) If $u = f(r)$, where $r = \sqrt{x^2 + y^2}$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$

(b) Trace the curve $x^2 y^2 = x^2 - a^2$.

*Or*

If $p = x\cos\theta + y\sin\theta$ touches the curve

$$\left(\frac{x}{a}\right)^{\frac{n}{n-1}} + \left(\frac{y}{b}\right)^{\frac{n}{n-1}} = 1$$

prove that $p^n = (a\cos\theta)^n + (b\sin\theta)^n$.

(c) Integrate :

$$\int \frac{dx}{3 + 2\sin x + \cos x}$$

*Or*

$$\int \frac{dx}{(1+x)\sqrt{1 + 2x - x^2}}$$

(d) Show that the volume of the solid produced by the revolution of the loop of the curve about the $x$-axis is

$$2\pi a^3\left(\log 2-\frac{2}{3}\right)$$

*Or*

Show that the volume of a right circular cone of height $h$ and base of radius $a$ is $\frac{1}{3}\pi a^2 h$.

4. Answer the following questions : 10×4=40

(a) (i) If $u$ is a homogeneous function of $x$ and $y$ of degree $n$ having continuous partial derivatives, prove that

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2 u=n(n-1)u$$

(ii) If $u=\cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, prove by using Euler's theorem on homogeneous function that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{2}\cot u$$

6+4=10

*Or*

Prove the following : 5+5=10

(i) $x^2 y_{n+2} + (2n+1)\, x y_{n+1} + (n^2+1)\, y_n = 0$ where $y = a\cos(\log x) + b\sin(\log x)$.

(ii) $D^n(\tan^{-1} x) = (-1)^{n-1} \lfloor n-1 \sin^n \theta \sin n\theta$ where $\theta = \cot^{-1} x$.

(b) A curve is represented in polar coordinates $(r, \theta)$ by the equation $r = f(\theta)$. Show that radius of curvature at the point $P(r, \theta)$ on the curve is given by

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

where $r_1 = f'(\theta)$ and $r_2 = f''(\theta)$. Hence or otherwise show that if $\rho_1$ and $\rho_2$ are the radii of curvature at the extremities of any focal chord of the cardioid $r = a(1+\cos\theta)$, which passes through the pole, then $\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$. 5+5=10

*Or*

Define cusp, isolated points, single cusp and double cusp. Find the position and nature of the multiple points on the curve $x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0$.

4+6=10

(c) (i) If $I_{m,n} = \int \cos^m x \cos nx \, dx$, prove that

$$I_{m,n} = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n} I_{m-1,\, n-1}$$

(ii) If $U_n = \int_0^{\pi/2} \theta \sin^n \theta \, d\theta$ and $n > 1$, then prove that

$$U_n = \frac{n-1}{n} U_{n-2} + \frac{1}{n^2}$$ 5+5=10

(d) (i) Find the area included between the curve $y^2(a-x) = x^3$ and its asymptote.

(ii) Show that the surface area of the solid generated due to the revolution of the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$ about the $x$-axis is $\frac{6}{5}\pi a^2$. 5+5=10

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