2012

MATHEMATICS

(Major)

Paper : 1.1

(Algebra and Trigonometry)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

(Objective-type questions)

(*Marks* : 10)

1. Answer the following questions. Each question carries four answers. Choose the correct answer : 1×10=10

(a) The map $f:\mathbb{R}\to\mathbb{R}$ given by $f(x)=x^3$, $x\in\mathbb{R}$ is

(i) one-one not onto

(ii) onto not one-one

(iii) Both one-one and onto

(iv) Neither one-one nor onto

(b) Let S be the set of all integers. Given $a, b \in S$, define relation R on S by aRb, if $a - b$ is an even integer. Then

(i) R is symmetric but not reflexive

(ii) R is an equivalence relation

(iii) R is not transitive

(iv) R is symmetric but not reflexive and transitive

(c) $(Z, -)$ is not a group because

(i) Z is not closed w.r.t. $-$

(ii) Z is not associative w.r.t. $-$

(iii) $(Z, -)$ has no identity element and $(Z, -)$ is not closed

(iv) $(Z, -)$ has identity but inverses do not exist

(d) If a and x are two elements of a group G such that $axa = b$, then

(i) $x = aba^{-1}$

(ii) $x = a^{-1}ba$

(iii) $x = a^{-1}b^{-1}a$

(iv) $x = a^{-1}ba^{-1}$

(e) All the values of z such that $\sin z = 0$ are given by

(i) $\frac{n\pi}{2}, n \in z$ (ii) $\frac{n\pi}{3}, n \in z$

(iii) $n\pi, n \in z$ (iv) $\frac{n\pi}{4}, n \in z$

(f) If n is an integer, then $(1+i)^n + (1-i)^n = ?$

(i) $2^{n/2+1} \cos\frac{n\pi}{4}$

(ii) $2^{n/2+1} \sin\frac{n\pi}{4}$

(iii) $2^{n/2} \cos\frac{n\pi}{4}$

(iv) $2^{n/2-1} \sin\frac{n\pi}{4}$

(g) Given that $1+i$ is a root of the $x^4 + x^2 - 2x + 6 = 0$. The roots of the equation are

(i) $1+i,\ i-1,\ -1+\sqrt{2}i,\ 1+\sqrt{2}i$

(ii) $1+i,\ 1-i,\ -1+\sqrt{2}i,\ -1-\sqrt{2}i$

(iii) $1+i,\ -1+i,\ -1-\sqrt{2}i,\ 1+\sqrt{2}i$

(iv) $1+i, -1-i,\ -1+\sqrt{2}i,\ -1-\sqrt{2}i$

(h) The equation $x^4 + 2x^2 + 3x - 1 = 0$ has

(i) three real roots and one complex root

(ii) two real roots and two complex roots

(iii) one real root and three complex roots

(iv) no real root

(i) The rank of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{pmatrix}$ is

(i) 1 (ii) 3

(iii) 2 (iv) 0

(j) Choose the correct statement :

(i) Product of two non-zero matrices is not necessarily non-zero

(ii) If $AB = 0$, BA is necessarily zero matrix

(iii) If $A \neq I$, A^2 is necessarily I

(iv) If $A \neq 0$, A^2 is necessarily 0

SECTION—B

(Very short-type questions)

(*Marks* : 10)

2. Give brief answer to the following questions :

$2 \times 5 = 10$

(a) Construct two partitions of the set $\{1, 2, 3, 4, 5\}$.

(b) Let $*$ be an associative binary operation on S and let m, n be positive integers. Show that $a^m * a^n = a^{m+n}$ and $(a^m)^n = a^{mn}$.

(c) Expand $\cos^7 \theta$ in a series of cosines of multiples of θ.

(d) If $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the roots of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_n = 0 \quad (p_n \neq 0)$$

find the value of $\sum \frac{1}{\alpha_1}$.

(e) For any square matrix A, show that

$$\text{adj } A' = (\text{adj } A)'$$

SECTION—C

(Short answer-type questions)

(*Marks* : 20)

3. Answer any *four* parts : 5×4=20

(a) Let $f : S \to T$ and $g : T \to U$. Prove that $f \circ g$ is onto if each of f and g is onto and $f \circ g$ is one-to-one if each of f and g is one-to-one.

(b) If H and K are finite subgroups of G of order $O(H)$ and $O(K)$ respectively, prove that

$$O(HK) = \frac{O(H) \cdot O(K)}{O(H \cap K)}$$

(c) Deduce the Gregory's series

$$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \cdots \text{ to } \infty$$

where $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$.

(d) Solve by Cardan's method

$$x^3 + 6x^2 + 9x + 4 = 0$$

(e) If $\cosh u = \sec\theta$, prove that

$$u = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) \quad \text{and} \quad \theta = \tan^{-1}(\sinh u)$$

(f) For the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{pmatrix}$$

find non-singular matrices P and Q such that PAQ is the normal form. Find the rank of A.

SECTION—D

(*Marks* : 40)

Answer *either* (a) and (b) *or* (c) and (d) :

4. (a) If $f : X \to Y$ and $g : Y \to Z$ be one-one and onto maps, then prove that $g \circ f$ is invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

(b) Describe the equivalence relation corresponding to the following partition of $\mathbb{Z}$:

$$\{\cdots, -8, -4, 0, 4, 8, \cdots\} \cup \{\cdots, -7, -3, 1, 5, \cdots\} \cup \{\cdots, -6, -2, 2, 6, \cdots\} \cup \{\cdots, -5, -1, 3, 7, \cdots\}$$

5+5=10

Or

(c) Prove that an injective mapping from a finite set to itself is bijective.

(d) Show that f is invertible if and only if f is bijective. 5+5=10

5. (a) If G is a group such that $(ab)^n = a^n b^n$ for three consecutive integers, then prove that $ab = ba$.

(b) Let G be a group such that $(ab)^2 = (ba)^2$ for all $a, b \in G$. Suppose G also has the property that $x^2 = e$, $x \in G$ implies $x = e$. Prove that G is Abelian. 5+5=10

Or

(c) Show that a necessary and sufficient condition for a non-empty subset H of a group G to be a subgroup is

$$a \in H,\ b \in H \Rightarrow ab^{-1} \in H$$

(d) Show that a subgroup H of a group G is a normal subgroup of G if and only if each left coset of H in G is a right coset of H in G. 5+5=10

6. (a) If $\cosh(u+iv) = x+iy$, prove that

$$\frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = 1 \text{ and } \frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = 1$$

(b) Separate $\tan^{-1}(x+iy)$ into real and imaginary parts. 5+5=10

Or

(c) Solve the equation

$$x^4 - 2x^3 - 21x^2 + 22x + 40 = 0$$

whose roots are in arithmetic progression.

(d) Transform the equation

$$x^3 - 6x^2 + 5x + 8 = 0$$

into another in which the second term is missing. 5+5=10

7. (a) Investigate the values of λ and μ so that the equations

$$2x+3y+5z=9,\ 7x+3y-2z=8,\ 2x+3y+\lambda z=\mu$$

have (i) no solution, (ii) a unique solution and (iii) infinite number of solutions.

(b) Show that every square matrix can be uniquely expressed as a sum of a Hermitian matrix and a skew-Hermitian matrix. 5+5=10

Or

(c) Prove that the necessary and sufficient condition for a square matrix A to possess an inverse is that $|A| \neq 0$.

(d) If A and B are n-rowed square matrices, prove that

$$\text{adj } AB = (\text{adj } B) \cdot (\text{adj } A)$$ 5+5=10

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